Random matrices and Gaussian multiplicative chaos

Nick Simm

Mathematics Institute, University of Warwick

Joint work with Gaultier Lambert and Dmitry Ostrovsky.

Optimal Point Configurations and Orthogonal Polynomials

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- The eigenvalues $e^{i\theta_1}, \ldots, e^{i\theta_N}$ form a determinantal point process with kernel

$$K_N(\theta, \phi) = \sum_{j=0}^{N-1} p_j(\theta) p_j(\phi) = \frac{\sin(N(\theta - \phi)/2)}{\sin((\theta - \phi)/2)}$$

where $p_j(\theta) = e^{ij\theta}$.

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  where $p_j(\theta) = e^{ij\theta}$.
- Problem: limit theorems for $P_N(\theta) = \det(U_N - e^{i\theta})$ as $N \to \infty$.

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- An interesting example of a log-correlated Gaussian field. E.g. how to compute

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- Using these ideas, it has been conjectured and *partially* proved that as $N \to \infty$

$$M_N^* = \log(N) - \frac{3}{4} \log(\log(N)) + (G_1 + G_2)/2 + o(1)$$

where $G_{1,2}$ are standard independent Gumbel variables. (Fyodorov and Keating '12, Arguin, Belius, Bourgade '15, Paquette and Zeitouni '16, Chaibbi, Madaule and Najnudel '16)

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Theorem (Hughes, Keating and O'Connell '01)

Let $\{Z_j\}_{j=1}^{\infty}$ be i.i.d. standard complex Gaussian random variables. Then

$$V_N(\theta) := \log |P_N(\theta)| \xrightarrow{d} V(\theta) := \frac{1}{2} \sum_{k=1}^{\infty} \frac{e^{i\theta}}{\sqrt{k}} Z_k + \mathrm{c.c.}$$

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Key properties of $V(\theta)$:

- V is *Gaussian* and mean zero $\mathbb{E}(V(\theta)) = 0$.
- Logarithmic correlations:

$$\mathbb{E}(V(\theta)V(\phi)) = \frac{1}{2}\mathrm{Re}\left\{\sum_{j=1}^{\infty} \frac{e^{ik(\theta-\phi)}}{k}\right\} = -\frac{1}{2}\log|e^{i\theta} - e^{i\phi}|$$

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Conclusion: Limit $V(\theta)$ is a *distribution valued object*.

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The measure $\mu^{(\gamma)}$ is defined by a *renormalization procedure* $V_\epsilon = V * \phi_\epsilon$.

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This limit *defines* the measure $\mu^{(\gamma)}$ which is called *Gaussian multiplicative chaos* (GMC).

Properties of measures $\mu^{(\gamma)}$

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- Power law spectrum (multifractality): For $0 \leq q\gamma^2 < 2$ we have
$$\mathbb{E}(\mu^{(\gamma)}[0, r]^q) = C_q r^{\xi(q)}$$
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- The distribution of $\mu^{(\gamma)}$ near $\gamma = \gamma_c$ is believed to be closely related to statistics of $\max_{|z|=1}|P_N(z)|$.

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$$\mathbb{E}(\mu^{(\gamma)}(D)^2) = \int_{D\times D} |e^{i\theta} - e^{i\phi}|^{-\gamma^2/2} d\theta \, d\phi < \infty$$

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Theorem (Webb '15)

Consider

$$\mu_N^{(\gamma)}(D) = \frac{\int_D |P_N(\theta)|^\gamma \, d\theta}{\mathbb{E} \int_D |P_N(\theta)|^\gamma \, d\theta}$$

Then for any $\gamma < \sqrt{2}$ *we have*

$$\mu_N^{(\gamma)} \xrightarrow{d} \mu^{(\gamma)}, \qquad N \to \infty$$

where $\mu^{(\gamma)}$ *is the same measure constructed from Kahane's theory.*

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Instead of $V_N(\theta)$, we consider *counting statistics*

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In reality, we consider a slightly smoother version:

$$X_{N,\epsilon_N}(\theta) = \sum_{j=1}^{N} (\chi_{J(\theta)} * \phi_\epsilon)(N^\alpha \theta_j), \qquad \phi_\epsilon(\theta) = \frac{1}{\epsilon}\phi\left(\frac{\theta}{\epsilon}\right)$$

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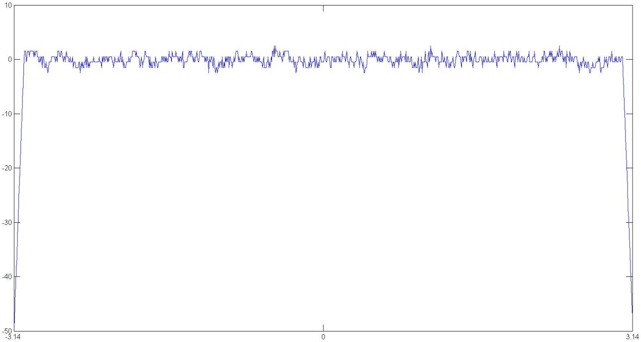
We will study the field $X_{N,\epsilon}$ with *mollifying scale* $\epsilon \to 0$ depending on N.

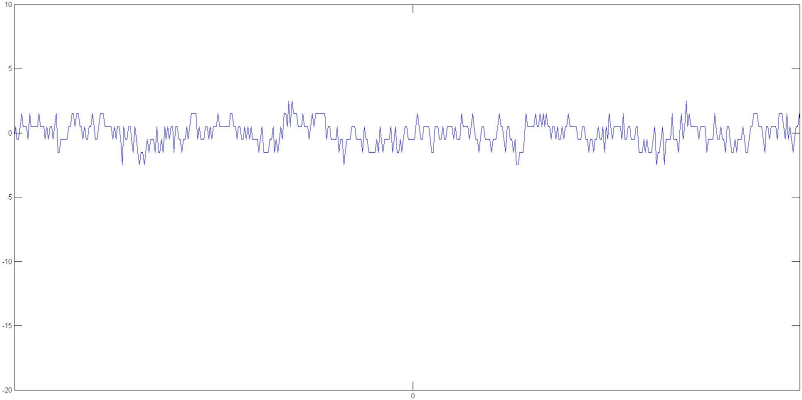
A realization of the process

A plot of the process $\tilde{X}_N(u) := X_N(u) - \mathbb{E}(X_N(u))$ and $N = 3000, \alpha = 0$.



A realization of the process

A plot of the process $\tilde{X}_N(u) := X_N(u) - \mathbb{E}(X_N(u))$, $u \in [-\pi, \pi)$ and $N = 3000$, $\alpha = 0$.



(Zoomed in around the origin $u \in (-0.2, 0.2)$)

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Theorem (Lambert, Ostrovsky, S' 2016)

Suppose $0 < \alpha < 1$ *and* $\epsilon_N \to 0$ *such that* $\epsilon_N^{-1} N^{\alpha-1} \to 0$. *Then for every* $\gamma < 2$ *we have*

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Thus we go beyond the L^2 bounds $\gamma < \sqrt{2}$ to establish convergence in the full phase $\gamma < 2$.

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Let f be a smooth function with rapid decay. Then

$$\sum_{j=1}^{N} f(\theta_j N^{\alpha}) - \mathbb{E}\left[\sum_{j=1}^{N} f(\theta_j N^{\alpha})\right] \xrightarrow{d} \mathcal{N}(0, \sigma^2(f))$$

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where

$$\sigma^2(f) = \|f\|_{H^{1/2}}^2 = \int_{-\infty}^{\infty} |k||\hat{f}(k)|^2 \, dk$$
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However, if $\epsilon \equiv \epsilon_N$ then $\mathrm{Var}(X_{N,\epsilon_N}(u)) \sim \frac{1}{2}\log(\epsilon_N^{-1})$ diverges...

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Lemma

Suppose $0 < \alpha < 1$ and $\epsilon_N \to 0$ such that $\epsilon_N^{-1} N^{\alpha-1} \to 0$. Then for any finite sequence of points $u_1, \ldots, u_q$, we have

$$\mathbb{E}\left(\exp\left(\sum_{j=1}^{q} \alpha_j \tilde{X}_{N,\epsilon_N}(u_j)\right)\right) = \exp\left(\frac{1}{2}\left\|\sum_{j=1}^{q} \alpha_j(\chi_{J(u_j)} * \phi_{\epsilon_N})\right\|_{H^{1/2}}^2\right)(1 + o(1))$$

as $N \to \infty$. The error term is uniform in $u_1, \ldots, u_q$ varying in compact subets of $\mathbb{R}$.

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Case $q = 2$ quite easily gives the L^2-phase:

$$\mu_{N,\epsilon_N} = \color{red}{\mu_{N,\epsilon}} + \color{blue}{(\mu_{N,\epsilon_N} - \mu_{N,\epsilon})}$$

Uniformity above allows precise computation of the *second moment* of the blue term, provided $\gamma < \sqrt{2}$.

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Theorem (Borodin-Okounkov formula (2000))

Let F be a 2π-periodic function. Then the following identity holds:

$$T_N[F] := \det\{\hat{F}_{k-j}\}_{k,j=0..N-1} = \exp\left(N\widehat{\log F_0} + \sum_{k=1}^{\infty} |k| |\widehat{\log F}_k|^2\right) \times \det(I - R_N H(c) H(c)^{\dagger} R_N)$$

where R_N are projections on $\{N+1, N+2, \ldots\}$ and $H(c) = \{\hat{c}_{j+k-1}\}_{j,k=1}^{\infty}$ where

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We prove that the last determinant is close to 1, uniformly as $N \to \infty$.

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Idea is to study γ*-thick points* (see *e.g.* Berestycki '15):

$$P_{c,\tau} := \{u \in [-c, c] : \tilde{X}_{N,\epsilon_N}(u) > \tau \log(\epsilon_N^{-1})\}$$

Note that $P_{c,\tau}$ are the points where the field fluctuates above its maximum.

We show that for any $\tau > \gamma$, the mass $\mu^{\gamma}_{N,\epsilon_N}(P_{c,\tau})$ converges to zero in L^1.

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We are guided by Berestycki's calculation, adapted to our 'almost Gaussian' setting.

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Thank you.