Orthogonal polynomials and zeros of optimal approximants

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Our basics

Definition

Let $\omega_0 = 1$, $\omega_k > 0$, and $\lim \frac{\omega_k}{\omega_{k+1}} = 1$. Then

$$H^2_\omega = \{f \in Hol(\mathbb{D}) : f(z) = \sum_{k\in\mathbb{N}} a_k z^k, ||f||^2_\omega = \sum_{k=0}^{\infty} |a_k|^2 \omega_k < \infty\}.$$

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f regular $+ Z(f) \cap \overline{\mathbb{D}} = \emptyset \Rightarrow$ cyclic $\Rightarrow Z(f) \cap \mathbb{D} = \emptyset$.

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$p_n^*(z) = \sum_{j=0}^n c_j z^j$ only solution to $Mc = b$ where

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- So we want to know about these polynomials! Today: where are their zeros? Which points of the plane may be zeros of such polynomials? (for a fixed space)

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In $\mathcal{H}^2$,

$$Z(p_n^*) = \{\overline{z_j}^{-1} : z_j \in Z(\varphi_n)\}.$$

In fact, these zero sets characterize cyclic functions.

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- If $\exists k, n : \omega_k > 4\omega_{k+n+1}$ then $f_{k,n}(z) = z^k T_n(\frac{1+z}{1-z})$ makes $z_0 \in \mathbb{D}$.

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where $\{P_k\}$ *is a particular family of monic orthogonal polynomials given by a recurrence relationship and* $\mathcal{J}_\omega$ *is the Jacobi matrix* $(\mathcal{J}_\omega)_{i,j} = \sqrt{\frac{\omega_j}{\omega_{j+1}}}$ *if* $|i-j| = 1; 0$*, o.w.*

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- With other classical and modern results on orthogonal polynomials and Jacobi matrices + some work on regularity of some measures $\Rightarrow$ our Thm.

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We were able to compute explicitly $\|\mathcal{J}_\omega\|$ for the following:

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Theorem

β, ω *as above.* $\|\mathcal{J}_\omega\| = \frac{\beta+3}{\sqrt{\beta+2}}$, $f^*(z) = \left(1 - \frac{z}{\sqrt{\beta+2}}\right)^{-(\beta+3)}$

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- For the solution to be analytic in $\mathbb{D}$ we need one zero of $(1 - \lambda z + z^2)$ to match $\lambda/3$. This gives the value of λ. Then solve everything else.

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- Many more problems! Ask me!

Gracias!