Bivariate orthogonal polynomials, 2D Toda lattices and Lax-type pairs

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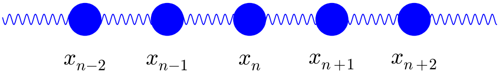
Join work with Cleonice F. Bracciali, Universidade Estadual Paulista (Brazil)

Contents

Contents

Toda lattice (Morikazu Toda, 1967)

- A Toda lattice is a simple model for a one-dimensional crystal in solid state physics



- Oscillations of a infinite system of points $\{x_n,\ n \geq 0\}$ joined by spring masses
- Interaction between the masses

$$\ddot{x}_n = e^{x_{n-1}-x_n} - e^{x_n - x_{n+1}}$$

where $\dot{y}(t) = dy(t)/dt$

Graphics by Miguel Piñar, made with SAGE.

Toda lattice

$$\ddot{x}_n = e^{x_{n-1}-x_n} - e^{x_n - x_{n+1}}$$

Transformation of the equation

$$d_n(t) = \dot{x}_n, \qquad c_n(t) = e^{x_{n-1}-x_n}$$

Toda system

$$\dot{d}_n(t) = c_n(t) - c_{n+1}(t)$$
$$\dot{c}_n(t) = c_n(t)\left[d_{n-1}(t) - d_n(t)\right]$$

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An explicit solution for the above system is given by the coefficients of the *three term recurrence relation* for (univariate) monic orthogonal polynomials associated with a modification of the measure

Relation with orthogonal polynomials

- Let $e^{-x\,t}\,d\mu(x)$ be a measure such that

$$\int_{\mathbb{R}} x^n\, e^{-x\,t}\, d\mu(x) < +\infty \quad n \in \mathbb{N} \qquad t \geq 0$$

Relation with orthogonal polynomials

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- $\{P_n(x,t)\}_{n\geq 0}$ associated monic orthogonal polynomial sequence

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- $\{P_n(x,t)\}_{n\geq 0}$ associated monic orthogonal polynomial sequence
- Three term recurrence relation: $P_{-1}(x,t) = 0$, $P_0(x,t) = 1$

$$x\, P_n(x,t) = P_{n+1}(x,t) + d_n(t)\, P_n(x,t) + c_n(t)\, P_{n-1}(x,t) \quad n \geq 0$$

where

$$d_n(t) = \frac{\int_{\mathbb{R}} x P_n(x,t)^2 e^{-x\,t} d\mu(x)}{\int_{\mathbb{R}} P_n(x,t)^2 e^{-x\,t} d\mu(x)}, \quad n \geq 0$$

$$c_n(t) = \frac{\int_{\mathbb{R}} P_n(x,t)^2 e^{-x\,t} d\mu(x)}{\int_{\mathbb{R}} P_{n-1}(x,t)^2 e^{-x\,t} d\mu(x)}, \quad n \geq 1$$

Relation with orthogonal polynomials

Theorem

$\{c_n(t)\}_{n\geq 1}$ and $\{d_n(t)\}_{n\geq 0}$ are solutions of the Toda lattice system

$$\dot{d}_n(t) = c_n(t) - c_{n+1}(t)$$
$$\dot{c}_n(t) = c_n(t)\left[d_{n-1}(t) - d_n(t)\right]$$

- M. E. H. Ismail, Classical and quantum orthogonal polynomials in one variable, Encyclopedia of Mathematics and its Applications 98. Cambridge University Press, 2005.
- F. Peherstorfer, On Toda lattices and orthogonal polynomials, J. Comput. Appl. Math., 133 (2001), 519-534.

Relation with orthonormal polynomials

If $\{Q_n(x,t)\}_{n\geq 0}$ is the associated orthonormal PS, then

$$x\,Q_n(x,t) = a_n(t)\,Q_{n+1}(x,t) + d_n(t)\,Q_n(x,t) + a_{n-1}(t)\,Q_{n-1}(x,t)$$

Toda equations

$$\begin{aligned} \dot{d}_n(t) &= a_{n-1}^2(t) - a_n^2(t) \\ \dot{a}_n(t) &= \frac{a_n(t)}{2}\,[d_n(t) - d_{n+1}(t)] \end{aligned}$$

Lax pair

Toda equations for orthonormal polynomials

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Lax pair

$$\dot{\mathcal{L}} = [\mathcal{L}, \mathcal{B}] = \mathcal{L}\,\mathcal{B} - \mathcal{B}\,\mathcal{L}.$$

where

$$\mathcal{L} = \begin{pmatrix} d_0 & a_0 & & & & \\ a_0 & d_1 & a_1 & & & \\ & a_1 & d_2 & \ddots & & \\ & & \ddots & \ddots & a_{n-1} & \\ & & & a_{n-1} & d_n & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}; \mathcal{B} = \frac{1}{2}\begin{pmatrix} 0 & a_0 & & & & \\ -a_0 & 0 & a_1 & & & \\ & -a_1 & 0 & \ddots & & \\ & & \ddots & \ddots & a_{n-1} & \\ & & & -a_{n-1} & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}$$

Lax-Nakamura pair

For monic OP, Toda equations

$$\dot{d}_n(t) = c_n(t) - c_{n+1}(t),$$
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can be expressed in the form

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Lax-Nakamura pair

$$\dot{\mathcal{N}} = [\mathcal{N}, \mathcal{C}] = \mathcal{N}\,\mathcal{C} - \mathcal{C}\,\mathcal{N}.$$

where

$$\mathcal{N} = \begin{pmatrix} d_0 & 1 & & & & \\ c_1 & d_1 & 1 & & & \\ & c_2 & d_2 & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & c_n & d_n & \ddots \\ & & & & \ddots & \ddots \end{pmatrix} \qquad \mathcal{C} = \begin{pmatrix} 0 & & & & & \\ -c_1 & 0 & & & & \\ & -c_2 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & -c_n & 0 & \\ & & & & \ddots & \ddots \end{pmatrix}$$

2D Toda lattices

- Is it possible to extend Toda lattices as above to 2D?

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  - Two independent time variables t and s
  - Two dependent space variables x and y

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Question

There exist 2D Toda systems related with bivariate orthogonal polynomials?

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Question

There exist 2D Toda systems related with bivariate orthogonal polynomials?

- We chose the second way, and we relate 2D Toda lattices with the vector representation for bivariate polynomials

Contents

Bivariate orthogonal polynomials

Observe that $\#\{x^{n-k}\,y^k : 0 \le k \le n\} = n+1 > 1$

Bivariate orthogonal polynomials

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Polynomial system (PS)

Sequence of vectors $\{\mathbb{P}_n\}_{n\ge 0}$ of increasing size $n+1$, defined as

$$\mathbb{P}_n = \mathbb{P}_n(x,y) = (P_{n-k,k}(x,y)) = \begin{pmatrix} P_{n,0}(x,y) \\ P_{n-1,1}(x,y) \\ \vdots \\ P_{0,n}(x,y) \end{pmatrix}_{(n+1)\times 1}$$

where $\{P_{n-k,k}(x,y), 0 \le k \le n\}$ are polynomials of total degree n independent modulus Π_{n-1}

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where $\{P_{n-k,k}(x,y), 0 \le k \le n\}$ are polynomials of total degree n independent modulus Π_{n-1}

Monic PS $\quad P_{n-k,k}(x,y) = x^{n-k}\,y^k + \text{lower degree terms}, \quad 0 \le k \le n$

Π_m denotes the linear space of bivariate polynomials of degree $\le m$

Orthogonal polynomial system (OPS)

- $d\mu \equiv d\mu(x,y)$ a positive measure on a domain $\Omega \subset \mathbb{R}^2$ such that

$$\int_\Omega x^n\, y^m\, d\mu(x,y) < +\infty, \quad n,m \geq 0$$

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- A PS $\{\mathbb{P}_n\}_{n\geq 0} \equiv \{(P_{n,0}(x,y), P_{n-1,1}(x,y), \ldots P_{0,n}(x,y))^T\}_{n\geq 0}$ is orthogonal (OPS) if

$$\langle \mathbb{P}_n, \mathbb{P}_m^T\rangle = 0 \in \mathcal{M}_{(n+1)\times(m+1)}(\mathbb{R}), \quad m \neq n,$$
$$\langle \mathbb{P}_n, \mathbb{P}_n^T\rangle = H_n \in \mathcal{M}_{(n+1)\times(n+1)}(\mathbb{R}),$$

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- mutually orthogonal if H_n is diagonal / orthonormal if $H_n = I_{n+1}$

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- mutually orthogonal if H_n is diagonal / orthonormal if $H_n = I_{n+1}$
- There exists a unique monic OPS associated with $\langle\cdot,\cdot\rangle$

Three term relations (TTR)

Let $\{\mathbb{P}_n\}_{n\geq 0}$ be a monic OPS. There exist $D_{n,i} : (n+1) \times (n+1)$, and $C_{n,i} : (n+1) \times n$ such that

$$x\,\mathbb{P}_n = L_{n,1}\,\mathbb{P}_{n+1} + D_{n,1}\,\mathbb{P}_n + C_{n,1}\,\mathbb{P}_{n-1}$$
$$y\,\mathbb{P}_n = L_{n,2}\,\mathbb{P}_{n+1} + D_{n,2}\,\mathbb{P}_n + C_{n,2}\,\mathbb{P}_{n-1}$$

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$$y\,\mathbb{P}_n = L_{n,2}\,\mathbb{P}_{n+1} + D_{n,2}\,\mathbb{P}_n + C_{n,2}\,\mathbb{P}_{n-1}$$

where $\mathbb{P}_{-1} = 0$, $C_{-1,i} = 0$, and

$$L_{n,1} = \left(\begin{array}{ccc|c} 1 & & \bigcirc & 0 \\ & \ddots & & \vdots \\ \bigcirc & & 1 & 0 \end{array}\right) \quad L_{n,2} = \left(\begin{array}{c|ccc} 0 & 1 & & \bigcirc \\ \vdots & & \ddots & \\ 0 & \bigcirc & & 1 \end{array}\right)$$

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Moreover,

$$D_{n,1}\,H_n = \langle x\,\mathbb{P}_n, \mathbb{P}_n^T\rangle \qquad C_{n,1}\,H_{n-1} = H_n\,L_{n-1,1}^T$$
$$D_{n,2}\,H_n = \langle y\,\mathbb{P}_n, \mathbb{P}_n^T\rangle \qquad C_{n,2}\,H_{n-1} = H_n\,L_{n-1,2}^T$$

and

$$\operatorname{rank} C_{n,i} = n \qquad \operatorname{rank} C_{n+1} = (C_{n,1}|C_{n,2}) = n+1$$

Contents

Toda modification of the measure

- For $t \geq 0$, define $d\tilde{\mu} \equiv d\tilde{\mu}(x,y,t) = e^{-(x+y)t} d\mu(x,y)$ such that

$$\int_{\Omega} x^n \, y^m \, e^{-(x+y)t} \, d\mu(x,y) < +\infty, \quad n,m \geq 0$$

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- $\{\mathbb{P}_n(t)\}_{n\geq 0} \equiv \{\mathbb{P}_n(x,y,t)\}_{n\geq 0}$ the monic OPS associated with $d\tilde{\mu}$

$$\mathbb{P}_n(t) = (P_{n,0}(x,y,t), P_{n-1,1}(x,y,t), \ldots, P_{0,n}(x,y,t))^T$$

where, for $0 \leq k \leq n$,

$$P_{n-k,k}(t) \equiv P_{n-k,k}(x,y,t) = x^{n-k} y^k + \sum_{m=0}^{n-1} \sum_{i=0}^{m} a_{m-i,i}(t) \, x^{m-i} \, y^i$$

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- Obviously, $\mathbb{P}_n(0) = \mathbb{P}_n$, and $\dot{P}_{n-k,k} \in \Pi_{n-1}$
- The positive–definite matrix

$$H_n(t) = \langle \mathbb{P}_n(t), \mathbb{P}_n^T(t) \rangle,$$

also depends on t, and $H_n(0) = H_n$, for $n \geq 0$.

Toda modification of the measure: TTR

There exist $D_{n,i}(t) : (n+1) \times (n+1)$, and $C_{n,i}(t) : (n+1) \times n$ such that

$$x\,\mathbb{P}_n(t) = L_{n,1}\,\mathbb{P}_{n+1}(t) + D_{n,1}(t)\,\mathbb{P}_n(t) + C_{n,1}(t)\,\mathbb{P}_{n-1}(t)$$
$$y\,\mathbb{P}_n(t) = L_{n,2}\,\mathbb{P}_{n+1}(t) + D_{n,2}(t)\,\mathbb{P}_n(t) + C_{n,2}(t)\,\mathbb{P}_{n-1}(t)$$

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where

$$D_{n,1}(t)\,H_n(t) = \int_\Omega x\,\mathbb{P}_n(t)\mathbb{P}_n(t)^T\,e^{-(x+y)t}d\mu(x,y)$$
$$D_{n,2}(t)\,H_n(t) = \int_\Omega y\,\mathbb{P}_n(t)\mathbb{P}_n(t)^T\,e^{-(x+y)t}d\mu(x,y)$$
$$C_{n,1}(t)\,H_{n-1}(t) = H_n(t)\,L_{n-1,1}^T$$
$$C_{n,2}(t)\,H_{n-1}(t) = H_n(t)\,L_{n-1,2}^T$$

Remark. $L_{n,1}$ and $L_{n,2}$ are independent of t

Main results

Lemma

Let $D_n(t) = D_{n,1}(t) + D_{n,2}(t)$. For $n \geq 0$

$$\dot{H}_n(t) = -D_n(t)\, H_n(t)$$

Main results

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Theorem

For $n \geq 1$ and $i = 1, 2$,

(i) $\dot{C}_{n,i}(t) = C_{n,i}(t)\, D_{n-1}(t) - D_n(t)\, C_{n,i}(t)$

(ii) $\dot{D}_{n,i}(t) = C_n(t)\, L_{n-1,i} - L_{n,i}\, C_{n+1}(t)$

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Corollary

For $n \geq 1$,

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(ii) $\dot{D}_n(t) = C_n(t)\, L_{n-1} - L_n\, C_{n+1}(t)$

Main results

Two variables

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Main results

Two variables

(i) $\dot{C}_n(t) = C_n(t)\, D_{n-1}(t) - D_n(t)\, C_n(t)$

(ii) $\dot{D}_n(t) = C_n(t)\, L_{n-1} - L_n\, C_{n+1}(t)$

One variable

(i) $\dot{c}_n(t) = c_n(t)\, [d_{n-1}(t) - d_n(t)]$

(ii) $\dot{d}_n(t) = c_n(t) - c_{n+1}(t)$

2D Toda lattice

- Oscillations of a mesh of particles on $\mathbb{R}^2$ $\{(x_i, y_j) : i, j \geq 0\}$

2D Toda lattice

- Oscillations of a mesh of particles on $\mathbb{R}^2$ $\{(x_i, y_j) : i, j \geq 0\}$
- For $n \geq 0$, define the $(n+1) \times (n+1)$ matrices

$$\mathcal{X}_{n,1}(t) = \begin{pmatrix} x_0 & x_0 & \cdots & x_0 \\ x_1 & x_1 & \cdots & x_1 \\ \vdots & \vdots & & \vdots \\ x_n & x_n & \cdots & x_n \end{pmatrix} \qquad \mathcal{X}_{n,2}(t) = \begin{pmatrix} y_0 & y_1 & \cdots & y_n \\ y_0 & y_1 & \cdots & y_n \\ \vdots & \vdots & & \vdots \\ y_0 & y_1 & \cdots & y_n \end{pmatrix}$$

2D Toda lattice

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- For $n \geq 0$, define the $(n+1) \times (n+1)$ matrices

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and

$$\mathcal{X}_n(t) = \mathcal{X}_{n,1}(t) + \mathcal{X}_{n,2}(t) = \begin{pmatrix} \chi_{0,0} & \chi_{0,1} & \cdots & \chi_{0,n} \\ \chi_{1,0} & \chi_{1,1} & \cdots & \chi_{1,n} \\ \vdots & \vdots & & \vdots \\ \chi_{n,0} & \chi_{n,1} & \cdots & \chi_{n,n} \end{pmatrix}$$

where $\chi_{i,j} = x_i + y_j$, $i, j \geq 0$

2D Toda lattice

Interaction between the masses

$$\ddot{\mathcal{X}}_{n,1} = e^{-\mathcal{X}_n} L_{n-1}^T e^{\mathcal{X}_{n-1}} L_{n-1,1} - L_{n,1} e^{-\mathcal{X}_{n+1}} L_n^T e^{\mathcal{X}_n}$$

$$\ddot{\mathcal{X}}_{n,2} = e^{-\mathcal{X}_n} L_{n-1}^T e^{\mathcal{X}_{n-1}} L_{n-1,2} - L_{n,2} e^{-\mathcal{X}_{n+1}} L_n^T e^{\mathcal{X}_n}$$

2D Toda lattice

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$$\ddot{\mathcal{X}}_{n,2} = e^{-\mathcal{X}_n} L_{n-1}^T e^{\mathcal{X}_{n-1}} L_{n-1,2} - L_{n,2} e^{-\mathcal{X}_{n+1}} L_n^T e^{\mathcal{X}_n}$$

Summing both equations

$$\ddot{\mathcal{X}}_n = \ddot{\mathcal{X}}_{n,1} + \ddot{\mathcal{X}}_{n,2} = e^{-\mathcal{X}_n} L_{n-1}^T e^{\mathcal{X}_{n-1}} L_{n-1} - L_n e^{-\mathcal{X}_{n+1}} L_n^T e^{\mathcal{X}_n}$$

2D Toda lattice

Interaction between the masses

$$\ddot{\mathcal{X}}_{n,1} = e^{-\mathcal{X}_n} L_{n-1}^T e^{\mathcal{X}_{n-1}} L_{n-1,1} - L_{n,1} e^{-\mathcal{X}_{n+1}} L_n^T e^{\mathcal{X}_n}$$

$$\ddot{\mathcal{X}}_{n,2} = e^{-\mathcal{X}_n} L_{n-1}^T e^{\mathcal{X}_{n-1}} L_{n-1,2} - L_{n,2} e^{-\mathcal{X}_{n+1}} L_n^T e^{\mathcal{X}_n}$$

Summing both equations

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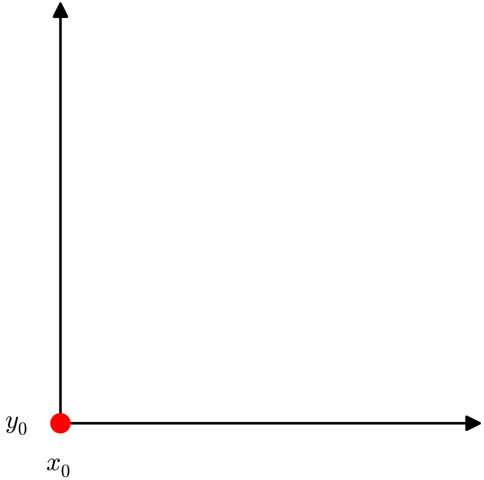
One variable

$$\ddot{x}_n = e^{x_{n-1}-x_n} - e^{x_n - x_{n+1}} = e^{-x_n} e^{x_{n-1}} - e^{-x_{n+1}} e^{x_n}$$

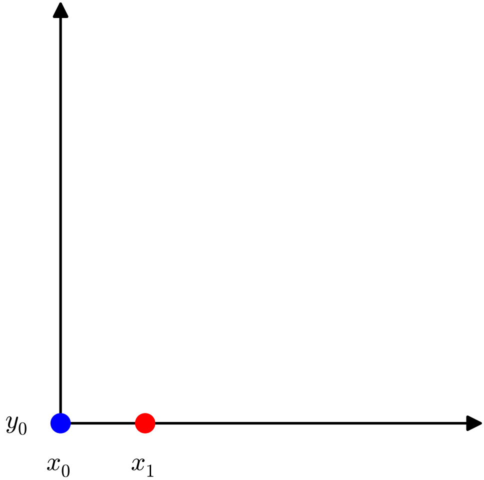
Note that $e^{-\mathcal{X}_n}$ is the exponential matrix of $-\mathcal{X}_n$

2D Toda lattice



Graphics by Miguel Piñar in SAGE

2D Toda lattice



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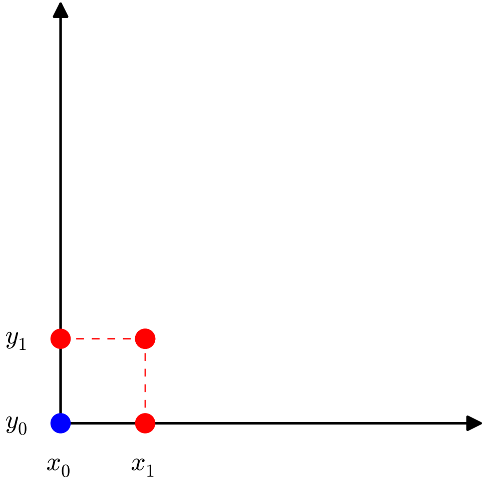

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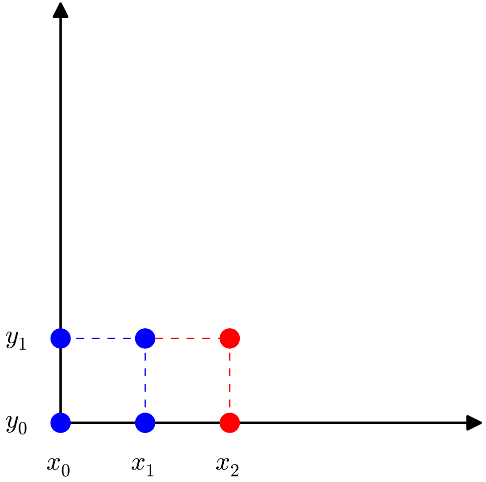

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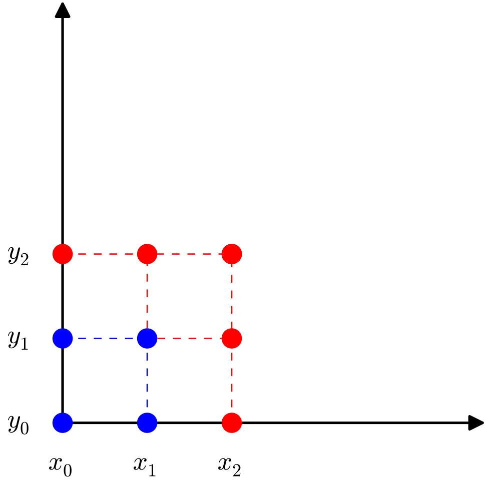

Graphics by Miguel Piñar in SAGE

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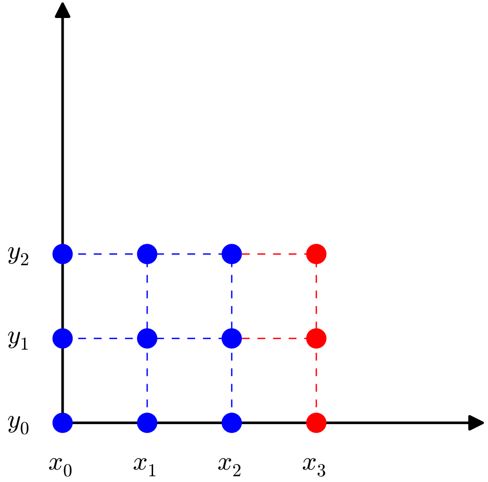

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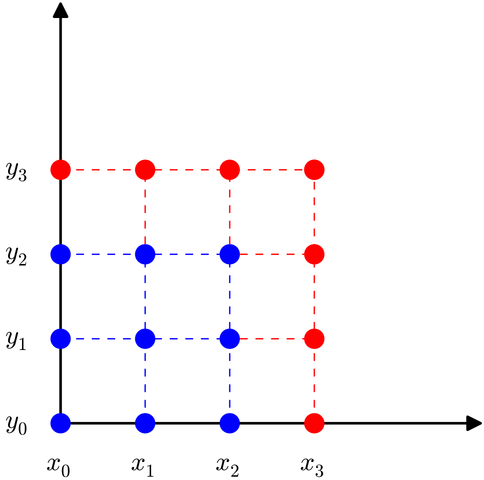
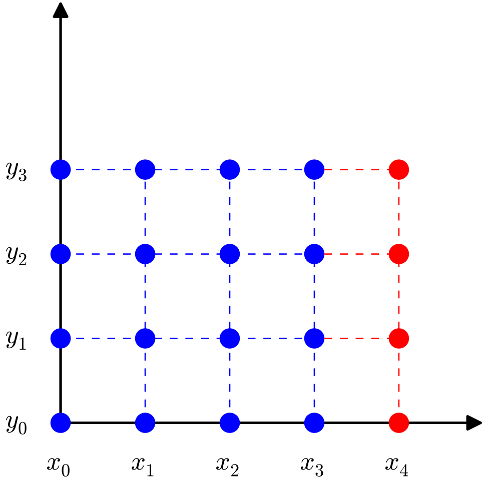
2D Toda lattice



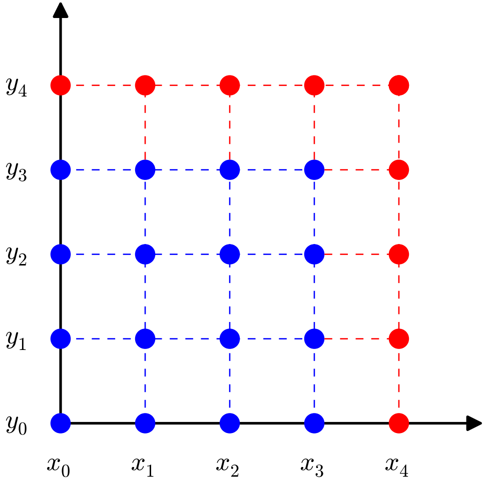
Graphics by Miguel Piñar in SAGE

2D Toda lattice



Graphics by Miguel Piñar in SAGE

2D Toda lattice



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2D Toda lattice

For $i = 1, 2$,

$$\ddot{\mathcal{X}}_{n,i} = e^{-\mathcal{X}_n} L_{n-1}^T e^{\mathcal{X}_{n-1}} L_{n-1,i} - L_{n,i} e^{-\mathcal{X}_{n+1}} L_n^T e^{\mathcal{X}_n}$$

- Transformation of the equation

$$\begin{aligned} D_{n,i}(t) &= \dot{\mathcal{X}}_{n,i} \\ C_{n,i}(t) &= e^{-\mathcal{X}_n} L_{n-1,i}^T e^{\mathcal{X}_{n-1}} \end{aligned}$$

- System: if $\mathcal{X}_{n,i}$ and $\dot{\mathcal{X}}_{n,i}$ commute then

$$\begin{aligned} \dot{D}_{n,i}(t) &= C_n(t) L_{n-1,i} - L_{n,i} C_{n+1}(t) \\ \dot{C}_{n,i}(t) &= C_{n,i}(t) D_{n-1}(t) - D_n(t) C_{n,i}(t) \end{aligned}$$

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Theorem

The coefficients of the three term relations for bivariate orthogonal polynomials constitute an explicit solution for the 2D Toda lattice

Contents

Block Lax-Nakamura pair

2D Toda lattice

$$\begin{aligned}\dot{D}_{n,i}(t) &= C_n(t)\,L_{n-1,i} - L_{n,i}\,C_{n+1}(t)\\ \dot{C}_{n,i}(t) &= C_{n,i}(t)\,D_{n-1}(t) - D_n(t)\,C_{n,i}(t)\end{aligned}$$

for the monic orthogonal polynomial system can be expressed as

Block Lax-Nakamura pair

2D Toda lattice

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for the monic orthogonal polynomial system can be expressed as

Block Lax-Nakamura pair

$$\dot{\mathcal{N}} = [\mathcal{N}, \mathcal{C}] = \mathcal{N}\,\mathcal{C} - \mathcal{C}\,\mathcal{N}$$

where

$$\mathcal{N} = \begin{pmatrix} D_0 & L_0 & & & & \\ C_1 & D_1 & L_1 & & & \\ & C_2 & D_2 & \ddots & & \\ & & \ddots & \ddots & L_{n-1} & \\ & & & C_n & D_n & \ddots \\ & & & & \ddots & \ddots \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} 0 & & & & & \\ -C_1 & 0 & & & & \\ & -C_2 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & -C_n & 0 & \\ & & & & \ddots & \ddots \end{pmatrix}$$

Block Lax-type pair

- $H_n(t)$ is symmetric and positive definite $\Rightarrow \exists\ H_n^{\frac{1}{2}}(t)$ symmetric and positive definite such that $H_n(t) = H_n^{\frac{1}{2}}(t)\, H_n^{\frac{1}{2}}(t)$

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- Then $\{\mathbb{Q}_n(t)\}_{n\geq 0} = \{H_n^{-\frac{1}{2}}(t)\, \mathbb{P}_n(t)\}_{n\geq 0}$ is an orthonormal PS

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- Then $\{\mathbb{Q}_n(t)\}_{n\geq 0} = \{H_n^{-\frac{1}{2}}(t)\, \mathbb{P}_n(t)\}_{n\geq 0}$ is an orthonormal PS

- Now, three term relations are given by

$$
\begin{aligned}
x\, \mathbb{Q}_n(t) &= A_{n,1}(t)\, \mathbb{Q}_{n+1}(t) + B_{n,1}(t)\, \mathbb{Q}_n(t) + A^T_{n-1,1}(t)\, \mathbb{Q}_{n-1}(t) \\
y\, \mathbb{Q}_n(t) &= A_{n,2}(t)\, \mathbb{Q}_{n+1}(t) + B_{n,2}(t)\, \mathbb{Q}_n(t) + A^T_{n-1,2}(t)\, \mathbb{Q}_{n-1}(t)
\end{aligned}
$$

where

$$
\begin{aligned}
A_{n,i}(t) &= H_n(t)^{-\frac{1}{2}} L_{n,i} H_{n+1}(t)^{\frac{1}{2}} = H_n(t)^{\frac{1}{2}} C^T_{n+1,i}(t) H_{n+1}(t)^{-\frac{1}{2}} \\
B_{n,i}(t) &= H_n(t)^{-\frac{1}{2}} D_{n,i}(t) H_n(t)^{\frac{1}{2}}
\end{aligned}
$$

Block Lax-type pair

Toda equations for monic PS

$$\begin{aligned}\dot{D}_{n,i}(t) &= C_n(t)\,L_{n-1,i} - L_{n,i}\,C_{n+1}(t)\\ \dot{C}_{n,i}(t) &= C_{n,i}(t)\,D_{n-1}(t) - D_n(t)\,C_{n,i}(t)\end{aligned}$$

become

Block Lax-type pair

Toda equations for monic PS

$$\begin{aligned} \dot{D}_{n,i}(t) &= C_n(t)\, L_{n-1,i} - L_{n,i}\, C_{n+1}(t) \\ \dot{C}_{n,i}(t) &= C_{n,i}(t)\, D_{n-1}(t) - D_n(t)\, C_{n,i}(t) \end{aligned}$$

become

2D Toda equations

$$\begin{aligned} \dot{A}_{n-1}^T &= A_{n-1}^T\, B_{n-1} - B_n\, A_{n-1}^T &+ \quad & \dot{H}_n^{-\frac{1}{2}}\, C_n\, H_{n-1}^{\frac{1}{2}} + H_n^{-\frac{1}{2}}\, C_n\, \dot{H}_{n-1}^{\frac{1}{2}} \\ \dot{B}_n &= A_{n-1}^T\, A_{n-1} - A_n\, A_n^T &+ \quad & \dot{H}_n^{-\frac{1}{2}}\, D_n\, H_n^{\frac{1}{2}} + H_n^{-\frac{1}{2}}\, D_n\, \dot{H}_n^{\frac{1}{2}} \end{aligned}$$

since the product of matrices is not commutative

Toda equations for orthonormal PS

Two variables

$$
\begin{aligned}
\dot{B}_n &= A_{n-1}^T A_{n-1} - A_n A_n^T &+& \quad \dot{H}_n^{-\frac{1}{2}} D_n H_n^{\frac{1}{2}} + H_n^{-\frac{1}{2}} D_n \dot{H}_n^{\frac{1}{2}} \\
\dot{A}_n^T &= A_n^T B_n - B_{n+1} A_n^T &+& \quad \dot{H}_{n+1}^{-\frac{1}{2}} C_{n+1} H_n^{\frac{1}{2}} + H_{n+1}^{-\frac{1}{2}} C_{n+1} \dot{H}_n^{\frac{1}{2}}
\end{aligned}
$$

Toda equations for orthonormal PS

Two variables

$$\dot{B}_n = A_{n-1}^T A_{n-1} - A_n A_n^T + \dot{H}_n^{-\frac{1}{2}} D_n H_n^{\frac{1}{2}} + H_n^{-\frac{1}{2}} D_n \dot{H}_n^{\frac{1}{2}}$$

$$\dot{A}_n^T = A_n^T B_n - B_{n+1} A_n^T + \dot{H}_{n+1}^{-\frac{1}{2}} C_{n+1} H_n^{\frac{1}{2}} + H_{n+1}^{-\frac{1}{2}} C_{n+1} \dot{H}_n^{\frac{1}{2}}$$

One variable

$$\dot{d}_n(t) = a_{n-1}^2(t) - a_n^2(t)$$

$$\dot{a}_n(t) = \frac{a_n(t)}{2} [d_n(t) - d_{n+1}(t)]$$

We must observe that, in one variable, $b_n(t) = d_n(t)$

Block Lax-type pair

$$\dot{\mathcal{L}} = [\mathcal{L} - \mathcal{K}, \mathcal{B}] + \dot{\mathcal{H}}^{-\frac{1}{2}} \, \mathcal{J} \, \mathcal{H}^{\frac{1}{2}} + \mathcal{H}^{-\frac{1}{2}} \, \mathcal{J} \, \dot{\mathcal{H}}^{\frac{1}{2}}$$

Block Lax-type pair

$$\dot{\mathcal{L}} = [\mathcal{L} - \mathcal{K}, \mathcal{B}] + \dot{\mathcal{H}}^{-\frac{1}{2}}\, \mathcal{J}\, \mathcal{H}^{\frac{1}{2}} + \mathcal{H}^{-\frac{1}{2}}\, \mathcal{J}\, \dot{\mathcal{H}}^{\frac{1}{2}}$$

$$\mathcal{L} = \begin{pmatrix} B_0 & A_0 & & & & \\ A_0^T & B_1 & A_1 & & & \\ & A_1^T & B_2 & \ddots & & \\ & & \ddots & \ddots & A_n & \\ & & & A_n^T & B_n & \ddots \\ & & & & \ddots & \ddots \end{pmatrix} \quad \mathcal{B} = \begin{pmatrix} 0 & A_0 & & & & \\ -A_0^T & 0 & A_1 & & & \\ & -A_1^T & 0 & \ddots & & \\ & & \ddots & \ddots & A_n & \\ & & & -A_n^T & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}$$

$$\mathcal{K} = \frac{1}{2} \begin{pmatrix} 0 & A_0 & & & & \\ A_0^T & 0 & A_1 & & & \\ & A_1^T & 0 & \ddots & & \\ & & \ddots & \ddots & A_n & \\ & & & A_n^T & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix} \qquad \mathcal{J} = \begin{pmatrix} D_0 & C_1^T & & & & \\ C_1 & D_1 & C_2^T & & & \\ & C_2 & D_2 & \ddots & & \\ & & \ddots & \ddots & C_n^T & \\ & & & C_n & D_n & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}$$

$$\mathcal{H}^{\frac{1}{2}} = \operatorname{diag}\{H_n^{\frac{1}{2}},\ n = 0, 1, \ldots\} \qquad \mathcal{H}^{-\frac{1}{2}} = \operatorname{diag}\{H_n^{-\frac{1}{2}},\ n = 0, 1, \ldots\}$$

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Thanks for your attention!!