Orthogonal polynomials, zeros and electrostatics

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Outline

- Let $\{p_n(x)\}_{n\geq 0}$ be a sequence of polynomials orthonormal (OPRL) with respect to a weight function $w(x) = \exp(-v(x))$ supported on an interval $[c, d] \subset R$, finite or infinite:
$$\int_c^d p_m(x)p_n(x)w(x)\,dx = \delta_{m,n}\,.$$
- Then $\{p_n\}_{n\geq 0}$ satisfies a three-term recurrence relation:
$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x), \quad n \geq 0,\, a_n > 0,\, p_{-1} = 0,\, p_0 = 1.$$
- Under certain assumptions on w, the orthonormal polynomials p_n also satisfy a difference-differential relation
$$p_n'(x) = A(x;n)p_{n-1}(x) - B(x;n)p_n(x)\,,$$
where $A(x,n)$, $B(x;n)$ are given in terms of w, a_n-s, and the values $p_n(c)$ and $p_n(d)$.
- A direct consequence of the above is that p_n satisfies also the second order linear differential equation
$$p_n''(x) - 2R(x;n)p_n'(x) + S(x;n)p_n(x) = 0\,, \quad \text{with}$$
$$R(x;n) = \frac{v'(x)}{2} + \frac{A'(x;n)}{2A(x;n)}\,,$$
$$S(x;n) = B'(x;n) - B(x;n)\frac{A'(x;n)}{A(x;n)} - B(x;n)[v'(x) + B(x;n)] + \frac{a_n}{a_{n-1}}A(x;n)A(x;n-1)\,.$$

From classical to semiclassical OP

M.E.H. Ismail, *An electrostatic model for zeros of general orthogonal polynomials*, Pacific J. Math. **193** (2000), 355–369.

- Ismail considers $\varphi'(x) = R(x;n)$, where $\varphi(x)$ is the external field

$$\varphi(x) = \frac{v(x)}{2} + \frac{\ln(k_n A(x;n))}{2} = \varphi_{\text{long}}(x) + \varphi_{\text{short}}(x)$$

- $\varphi(x)$ has two components: $\varphi_{\text{long}}(x)$ comes from the orthogonality weight $w(x) = \exp(-v(x))$, and Ismail called it the *long range potential*.
- The other $\varphi_{\text{short}}(x)$ is called *short range potential*, allows to give a further generalization of the electrostatic interpretation.
- Ismail proves that, under certain assumptions, the total energy of the system has a unique point of global minimum, **which is located at the vector constituted by the zeros of the orthogonal polynomial p_n.**
- $A(x;n)$ is responsible of the creation of "*ghost*" movable charges, as it was shown first in:

F.A. Grünbaum, *Variations on a theme of Heine and Stieltjes: An electrostatic interpretation of the zeros of certain polynomials*, J. Comput. Appl. Math. **99** (1998), 189–194.

Semiclassical OPRL

- Let us consider here a little bit more general situation. A generalized weight function (or linear functional) is *semiclassical* if it satisfies the *Pearson equation*

$$D(\phi w) = \psi w \,,$$

where ϕ, ψ are polynomials, with degree of $\psi \geq 1$, and D is the "derivative" operator (in the usual, but also possibly in a distributional sense).

- It is well known that for such a weight the corresponding orthogonal polynomials (called also *semiclassical*) satisfy a differential equation of the type referred above, where the coefficients $R(x; n)$ and $S(x; n)$ are rational functions.

- The classical-type orthogonal polynomials considered by Grünbaum in the above work are an example of a semiclassical family, but there are many more.

F. Marcellán, A. Martínez-Finkelshtein, P. Martínez-González, *Electrostatic models for zeros of polynomials: old, new, and some open problems*, J. Comput. Appl. Math. **207** (2007), no. 2, 258–272.

Outline

Orthogonal and Paraorthogonal polynomials on the unit circle

- Given an infinitely supported probability measure μ on the unit circle, one defines the OPUC $\{\Phi_n(z;\mu)\}_{n\geq 0}$ as the monic sequence of monic polynomials satisfying the Szegő recursion:

$$\Phi_{n+1}(z;\mu) = z\Phi_n(z;\mu) - \bar{\alpha}_n\Phi_n^*(z;\mu),$$

where $\alpha_n \in \mathbb{D} := \{z : |z| < 1\}$ and $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\bar{z})}$.

- To each μ on the unit circle, we can associate a sequence $\{\alpha_n\}_{n\geq 0}$ of corresponding *Verblunsky coefficients*.

- For the above μ, and a complex number β of modulus 1, one can defines **paraorthogonal polynomials** (POPUC) $\{\Phi_n(z;\beta;\mu)\}_{n\geq 0}$ as the sequence of polynomials given by

$$\Phi_n(z;\beta;\mu) := z\Phi_{n-1}(z;\mu) - \bar{\beta}\Phi_{n-1}^*(z;\mu).$$

- All the zeros of $\Phi_n(z;\beta;\mu)$ are simple and **lie on the unit circle**.

Paraorthogonal polynomials on the unit circle

- If $\tau \neq \beta$ are distinct complex numbers of modulus 1, then the zeros of $\Phi_n(z;\beta;\mu)$ and $\Phi_n(z;\tau;\mu)$ strictly interlace on the unit circle, i.e.if x and y are two zeros of $\Phi_n(z;\beta;\mu)$ and $[x,y]$ is the arc of the unit circle that runs from x to y in the counter-clockwise direction, then $[x,y]\setminus\{x,y\}$ contains a zero of $\Phi_n(z;\tau;\mu)$.

- Paraorthogonal polynomials **are not** orthogonal polynomials, but they often serve as an appropriate analog of OPRL in settings where the real line is replaced by the unit circle.

- The basic reference in this section will be:

B. Simanek, *An electrostatic interpretation of the zeros of paraorthogonal polynomials on the unit circle.* SIAM J. Math. Anal. **48** (3) (2016), 2250–2268.

Theorem 1 (B. Simanek, 2016)

Suppose $d\mu(\theta) = w(\theta)\frac{d\theta}{2\pi}$ is a probability measure on the unit circle, where w is continuous on $[0, 2\pi]$ (mod 2π) and differentiable on $(0, 2\pi)$ and let $\{\alpha_n\}_{n=0}^{\infty}$ be the corresponding sequence of Verblunsky coefficients. If $\beta \in \mathbb{C}$, then the POPUC polynomial $y(z) = \Phi_n(z;\beta)$ defined above solves the following differential equation on any domain including infinity or zero on which the coefficients are meromorphic:

$$0 = y''(z) + \left[\frac{1-n}{z} - \frac{h_n'(z;\beta;\beta)}{h_n(z;\beta;\beta)}\right] y'(z)$$
$$+ \left[\frac{W[h_n(z;\beta;\beta), h_n(z;-\beta;\beta)]}{2\bar{\beta} z h_n(z;\beta;\beta)} - \frac{1}{z}\left((n + zG_n(z))G_n(z) + J_n(z)(D_n(z) - n\alpha_{n-1})\right)\right] y(z),$$

where

$$G_n(z) := i\int_0^{2\pi} \frac{|\varphi_{n-1}^*(e^{i\theta})|^2 w'(\theta)}{(z - e^{i\theta})} \frac{d\theta}{2\pi},$$

$$D_n(z) := -iz\int_0^{2\pi} \frac{\varphi_{n-1}^*(e^{i\theta})^2 w'(\theta)}{(z - e^{i\theta})e^{in\theta}} \frac{d\theta}{2\pi}, \qquad J_n(z) := i\int_0^{2\pi} \frac{\varphi_{n-1}(e^{i\theta})^2 w'(\theta)}{(z - e^{i\theta})e^{i(n-2)\theta}} \frac{d\theta}{2\pi},$$

$$h_n(z;x;y) := \bar{x}(n(1 - \bar{y}\alpha_{n-1}) + zG_n(z) + \bar{y}D_n(z)) - z(J_n(z) - \bar{y}G_n(z)),$$

and $W[f, g]$ denotes the Wronskian of f and g.

Theorem 2 (B. Simanek, 2016)

Suppose μ is as in the above Theorem and $\tau \neq \beta$ are complex numbers. The polynomials $u(z) := \Phi_n(z;\beta,\mu)$ and $\upsilon(z) := \Phi_n(z;\tau,\mu)$ solve the following system of differential equations on any domain containing infinity or zero where the coefficients are meromorphic:

$$u'(z) = \upsilon(z)\left(\frac{h_n(z;\beta;\beta)}{z(\bar{\beta}-\bar{\tau})}\right) - u(z)\left(\frac{h_n(z;\tau;\beta)}{z(\bar{\beta}-\bar{\tau})}\right)$$
$$\upsilon'(z) = \upsilon(z)\left(\frac{h_n(z;\beta;\tau)}{z(\bar{\beta}-\bar{\tau})}\right) - u(z)\left(\frac{h_n(z;\tau;\tau)}{z(\bar{\beta}-\bar{\tau})}\right).$$

More precisely, we need to know exactly the coefficient of $n^{-d/2}$ to estimate the above expressions correctly.

- The main restriction in the applicability of the above two Theorems is the requirement that the measure is absolutely continuous and the weight is a continuous and differentiable function.

Electrostatic interpretation

- The main application of the above results is to prove that the zeros of certain families of POPUC are points satisfying an electrostatic equilibrium.

- Simanek considers the problem of creating an electric field that will keep identical charges at fixed points on the unit circle stationary. More precisely, assume a set $\{x_1, \ldots, x_n\} \subseteq \partial\mathbb{D}$.

- He proves a way to find a number m, a collection of points $\{a_i\}_{i=1}^{m} \subseteq \mathbb{C} \setminus \{x_1, \ldots, x_n\}$, and a set of real charges $\{q_i\}_{i=1}^{m}$ so that if a particle of charge $+1$ is placed at each x_j ($j = 1, \ldots, n$) and a particle of charge q_i is placed at a_i ($i = 1, \ldots, m$), then the total force on the particle at each x_j is zero ($j = 1, \ldots, n$).

Electrostatic interpretation

- In other words, if we have a collection of m identically charged particles all lying on a concentric circle, it can be showed a way to construct an electric field that will keep these particles stationary.

- The points $\{a_i\}_{i=1}^{m}$ and charges $\{q_i\}_{i=1}^{m}$ comprise the so called a *set of electric field generators.* The charged particles at $\{x_j\}_{j=1}^{n}$ would be called *mobile charges* and the charged particles at $\{a_i\}_{i=1}^{m}$ would be called *impurity charges*.

- It is important to keep in mind that the charged particles at the points $\{x_j\}_{j=1}^{n}$ interact with each other as well as with the electric field generators. With our motivation now clearly stated, we provide the following definitions:

Definition:

(i) Given a set of electric field generators $\{a_i\}_{i=1}^m$ and $\{q_i\}_{i=1}^m$, a set of points $\{x_j\}_{i=1}^n$ located on a smooth curve Γ is in Γ-normal electrostatic equilibrium if for each $j = 1, 2, \ldots, n$, the force at x_j is normal to Γ at x_j.

(ii) Given a set of electric fields generators $\{a_i\}_{i=1}^m$ and $\{q_i\}_{i=1}^m$, a set of points $\{x_j\}_{i=1}^n \subset \partial\mathbb{D}$ is in total electrostatic equilibrium if for each $j = 1, 2, \ldots, n$

$$\sum_{k=1, k\neq j} \frac{1}{x_j - x_k} + \sum \frac{q_i}{x_j - a_i} = 0.$$

- If $\Gamma \subset \partial\mathbb{D}$, then the Γ-normal electrostatic equilibrium can be rewritten as

$$\mathrm{Im}\left[x_j\left(\sum_{k=1, k\neq j} \frac{1}{x_j - x_k} + \sum \frac{q_i}{x_j - a_i}\right)\right] = 0, \quad j = 1, 2, \ldots, n.$$

- Total electrostatic equilibrium $\Longrightarrow$ $\partial\mathbb{D}$-normal electrostatic equilibrium, but the converse is not true in general. Indeed, n particles of identical nonzero charge located at the nth roots of i¡unit and subject to no external force are in $\partial\mathbb{D}$-normal electrostatic equilibrium but are not in total electrostatic equilibrium.

Main result: Theorem 3 (Simanek, 2016)

Given any collection of $n \geq 2$ distinct points $\{x_1, \ldots, x_n\} \subseteq \partial\mathbb{D}$, there exists a set of electric field generators so that the collection $\{x_1, \ldots, x_n\}$ is in total electrostatic equilibrium.
In fact, given any n distinct points $\{x_1, \ldots, x_n\} \subseteq \partial\mathbb{D}$, we can deduce an explicit algorithm for finding the electric field generators. We proceed as follows:

- Step 1: Define the measure μ_n on $\partial\mathbb{D}$ by

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{x_j},$$

  and define $\beta := (-1)^{n+1} \prod_{j=1}^{n} \bar{x}_j$.

- Step 2: Consider the Gram-Schmidt orthogonalization process for the linearly independent set $\{1, z, \ldots, z^{n-1}\}$ in $L^2(\partial\mathbb{D}, \mu_n)$ to get the sequence of orthonormal polynomials $\{1, \varphi_1(z;\mu_n), \ldots, \varphi_{n-1}(z;\mu_n)\}$.

Main result: Theorem 3 (Simanek, 2016)

- Step 3: Define the probability measure

$$d\nu_n := \frac{1}{|\varphi_{n-1}(e^{i\theta};\mu_n)|^2}\frac{d\theta}{2\pi}.$$

- Step 4: Calculate the quantity $h_n(z;\beta;\beta)$ for the measure ν_n in the domain $\{z : |z| > 1\}$. Indeed, it will be a rational function $S_1(z)/S_2(z)$ for some polynomials S_1 and S_2.

- Step 5: Place a particle of charge $-1/2$ at each zero of S_1, a particle of charge $+1/2$ at each zero of S_2, and a particle of charge $\frac{1}{2}(1-n)$ at zero.

Basic idea: To translate the equilibrium problem into a Lamé differential equation.

Example: Lebesgue Polynomials

- Let μ be Lebesgue measure on the circle. In this case $\alpha_{n-1} = 0$ and w is constant so $w' = 0$.

- Let us also assume $\beta = 1$, so that $\Phi_n(z;\beta) = z^n - 1$ and $h_n(z;\beta;\beta) = n$.

- With this choice, from Theorem 2 we see that particles of charge $+1$ located at the n^{th} roots of unity are in total electrostatic equilibrium when the external field is generated by a charge of $\frac{1}{2}(1-n)$ located at the origin.

Example: Chebyshev Polynomials

- Consider the measure

$$d\mu(\theta) = (1-\cos(\theta))\frac{d\theta}{2\pi} = \frac{|1-e^{i\theta}|^2}{2}\frac{d\theta}{2\pi}.$$

- For this measure, the Verblunsky coefficients satisfy $\alpha_n = -(n+2)^{-1}$.

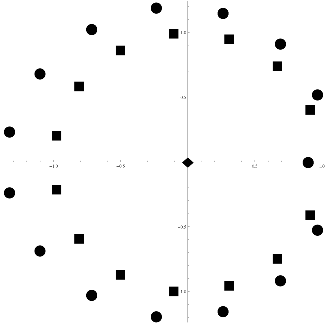
- The corresponding monic POPUC are $\Phi_n(z;-1) = (z^{n+1}-1)\big/(z-1)$, with zeros located at the $(n+1)^{st}$ roots of unity, up to $z=1$.

- $h_n(z;-1,-1) = \frac{P_{3,n}(z)}{P_{4,n}(z)}$,
  where
  $P_{3,n}(z) = -n(nz^{n+2} - (n+2)z^{n+1} + z + 1 = -n^2(z-1)\prod_{j=1}^{n+1}(z-p_{n,j})$
  $P_{4,n}(z) = (n+1)z^{n+1}(z-1)$

Example: Chebyshev Polynomials

- This means that particles of charge $+1$ located at the zeros of $\Phi_n(z;-1)$ are in $\partial\mathbb{D}$-normal electrostatic equilibrium when the external field is generated by a single particle of charge $+1$ located at 1.

- On the other hand, the particles of charge $+1$ located at the zeros of $\Phi_n(z;-1)$ are in total electrostatics equilibrium when the external field is generated by particles of charge $-1/2$ at each $p_{n,j}$ and a particle of charge $+1$ at the origin.



Outline

Exceptional orthogonal polynomials

- Exceptional orthogonal polynomials were introduced in the seminal work

D. Gómez-Ullate, N. Kamran , R. Milson, *An extended class of orthogonal polynomials defined by a Sturm-Liouville problem*, J. Math Anal. Appl. **359** (1) (2009) 352–367.

- Let $\{y_n\}_{n\in\sigma}$, $\sigma = \mathbb{N} - \{i_1, \ldots, i_m\}$ be a sequence of monic polynomials with $\deg y_n = n$, orthogonal with respect to a weight and, in addition, they are also eigenfunctions of a second order differential operator, i.e.

$$p(x)y_n'' + q(x)y_n' + r(x)y_n = \lambda_n y_n,$$

where p, q, r are rational functions.

- σ is a numerable set of $\mathbb{N}$ called the *degree sequence and* m is a number of missing integers in the degree sequence, known as the *codimension.*

- Concerning the location of zeros of exceptional orthogonal polynomials, the seminal reference is

D. Gómez-Ullate, F. Marcellán, R. Milson. *Asymptotic and interlacing properties of zeros of exceptional Jacobi and Laguerre polynomials*. J. Math. Anal. Appl. **399** (2013) 480–495.

Definition: weighted Fejér constants

- Next we analyze the electrostatic properties of zeros of Exceptional orthogonal polynomials. The basic reference in this section will be:

Á.P. Horváth, *The electrostatic properties of zeros of exceptional Laguerre and Jacobi polynomials and stable interpolation*, J. Approx. Theory **194** (2015), 87–107.

- Let $U_n := \{u_1, \ldots, u_n\}$ be any system of nodes on an interval I and $0 < w \in C^2(I)$ be a weight function on I.
- Let $\omega_{U_n}(x) := \prod_{k=1}^n (x - u_k)$
- The weighted Fejér constants on I with respect to U_n and w are:

$$C_k := C_{k,U_n,w} = \frac{\omega''_{U_n}}{\omega'_{U_n}}(u_k) + \frac{w'}{w}(u_k).$$

- We will investigate the local extrema of the energy function

$$T_w(u_1, \ldots, u_n) = \prod_{j=1}^n w(u_j) \prod_{1 \le i < j \le n} (u_i - u_j)^2.$$

- Basic idea: The differential equation of orthogonal polynomials, and its transformed version to a Schrödinger equation.

Lemma (Horváth, 2015)

Let $p_n(x) = \prod_{i=1}^{n}(x - \zeta_i)$, be a polynomial of degree n such that

$$p_n''(x) + M(x;n)p_n'(x) + N(x;n)p_n(x) = 0.$$

Let us assume that $M(x;n)$ is the logarithmic derivative of a function $w_n(x)$ which is smooth enough, that is

$$(\log w_n(x))' = M(x;n).$$

Let $T_{w_n}(u_1, \ldots, u_n)$ be the energy function with respect to w_n. Then for $i, j = 1, 2, \ldots$

$$\frac{\partial \log T_{w_n}(u_1,\ldots,u_n)}{\partial u_i}(\zeta_1,\ldots,\zeta_n) = C_{i,w_n,Z_n} = 0, \qquad \frac{\partial^2 \log T_{w_n}(u_1,\ldots,u_n)}{\partial u_i \partial u_j}(\zeta_1,\ldots,\zeta_n) = \frac{2}{(\zeta_i - \zeta_j)^2},$$

$$\frac{\partial^2 \log T_{w_n}(u_1,\ldots,u_n)}{\partial u_i^2}(\zeta_1,\ldots,\zeta_n) = -\frac{2}{3}\Phi(\zeta_i),$$

where $\Phi(x) := \Phi_{w_n}(x)$ is the coefficient of the transformed differential equation:

$$z_n''(x) + \Phi(x)z_n(x) = 0,$$

which is satisfied by

$$z_n(x) = p_n(x)\sqrt{w_n(x)}.$$

- Let $L_k^{(\alpha-1)}(x)$ be the k-th degree classical Laguerre polynomial of parameter $\alpha-1$.
- Let $\{L_{m,m+n}^{I,(\alpha)}\}_{n=0}^{\infty}$ denotes the sequence of exceptional Laguerre polynomials of the first kind with codimension $m\geq 1$.
- They are the orthogonal polynomials on $(0,\infty)$ with respect to the weight

$$\hat{w}_m^{(\alpha)} := \frac{x^{\alpha}e^{-x}}{S^2(x)}, \quad S(x) := s_m^{(\alpha-1)}(x) := L_m^{(\alpha-1)}(-x)$$

- $L_{m,m+n}^{I,(\alpha)}$ satisfies the following differential equation on $\mathbb{R}\diagdown\{0,-y_1,\ldots,-y_m\}$

$$y''(x)+\left(\frac{\alpha+1-x}{x}-\frac{2S'(x)}{S(x)}\right)y'(x)+\left(\frac{m+n}{x}-\frac{\alpha}{x}\frac{2S'(x)}{S(x)}\right)y(x)=0.$$

- **Lemma (Gómez Ullate et al, 2013):** $L^{I,(\alpha)}_{m,m+n}$ has $n+m$ simple zeros: n regular zeros $x^{(\alpha)}_{m,n,1},\ldots,x^{(\alpha)}_{m,n,n}\in(0,\ \infty)$ and m exceptional zeros $z^{(\alpha)}_{m,n,1},\ldots,z^{(\alpha)}_{m,n,m}\in(-\infty,0)$. Furthermore

$$\lim_{n\to\infty} n\,x^{(\alpha)}_{m,n,i}=\frac{\left(j^{(\alpha)}_i\right)^2}{4}$$

and as $n\to\infty$ the exceptional zeros of $L^{I,(\alpha)}_{m,m+n}$ converge to the zeros of $S(x)=L^{(\alpha-1)}_m(-x)$. Here $\{j^{(\alpha)}_i\}_{i\geq 1}$ is the increasing sequence of the positive zeros of the Bessel function of the first kind.

- **Theorem (Horváth, 2015):** Let $\alpha\geq 1$. Let us denote by $X_{m,n}=\{x_{m,n,1},\ldots,x_{m,n,n}\}$ the positive zeros of $L^{I,(\alpha)}_{m,m+n}$. $X_{m,n}$ is the Fekete set that is the unique set where the logarithmic energy function $(-\log T_\upsilon)$ takes its infimum under the external field represented by

$$\left(\upsilon^{(\alpha+1)}_{m,n}\right)^{\frac{1}{2(n-1)}},\quad \text{where } \upsilon^{(\alpha+1)}_{m,n}=\frac{x^{\alpha+1}e^{-x}P^2(x)}{S^2(x)},\quad P(x):=\prod_{i=1}^{m}(x-z_{m,n,i}).$$

X_m-Laguerre-(II) polynomials

- The type II exceptional Laguerre polynomials of codimension $m \geq 1$ are denoted by $\{L^{II,(\alpha)}_{m,m+n}\}_{n=0}^{\infty}$.
- Let us denote $S(x) := L_m^{(-\alpha-1)}(x)$ and let us assume that $\alpha > m-1$.
- It is known that S has no zeros in $[0,\ \infty)$ and $L^{II,(\alpha)}_{m,m+n}$-s are orthogonal on $(0,\infty)$ with respect to the weight
$$\hat{w}^{(\alpha)} := \frac{x^{\alpha}e^{-x}}{S^2(x)}$$
- $L^{II,(\alpha)}_{m,m+n}$ satisfies
$$y''(x) + \left(\frac{\alpha+1-x}{x} - \frac{2S'(x)}{S(x)}\right)y'(x) + \left(\frac{n-m}{x} - \frac{\alpha}{x}\frac{2S'(x)}{S(x)}\right)y(x) = 0.$$
- **Lemma (Gómez Ullate et al, 2013):** $L^{II,(\alpha)}_{m,m+n}$ has $n+m$ simple zeros, n regular zeros $x^{(\alpha)}_{m,n,1},\dots,x^{(\alpha)}_{m,n,n} \in (0,\ \infty)$ and 0 or 1 negative zero according to whether m is even or odd. Furthermore, as $n\to\infty$ the exceptional zeros of $L^{II,(\alpha)}_{m,m+n}$ converge to the zeros of $S(x) = L_m^{(-\alpha-1)}(x)$ and $\lim_{n\to\infty} n x^{(\alpha)}_{m,n,i} = \frac{\left(j_i^{(\alpha)}\right)^2}{4}$.

X_m-Laguerre-(II) polynomials

- We can write $L^{II,(\alpha)}_{m,m+n} = P(x)q(x)$, where $P(x)$ is a polynomial of degree m with 0 or 1 real zero.

- We can examine the properties of the regular zeros of $L^{II,(\alpha)}_{m,m+n}$, i.e. the zeros of q. One can write

$$q''(x) + \left(M(x;n) + 2\frac{P'}{P}(x)\right) q'(x) + \left(N(x;n) + \frac{P''}{P}(x) + M(x;n)\frac{P'}{P}(x)\right) q(x) = 0,$$

where $M(x;n)$ and $N(x;n)$ are the coefficients in the former differential equation for $L^{II,(\alpha)}_{m,m+n}$

$$M(x;n) = \tfrac{\alpha+1-x}{x} - \tfrac{2S'(x)}{S(x)}, \qquad N(x;n) = \tfrac{n-m}{x} - \tfrac{\alpha}{x}\tfrac{2S'(x)}{S(x)}.$$

- **Theorem (Horváth, 2015):** Let $\upsilon(x) = \upsilon^{(\alpha+1)}_{m,n}(x) = \frac{x^{\alpha}e^{-x}P^2(x)}{S^2(x)}$. If $\alpha > m-1$, and if n is large enough, then the set of regular zeros of $L^{II,(\alpha)}_{m,m+n}$ is the unique set on which the logarithmic energy function takes its minimum under the external field

$$\left(\upsilon^{(\alpha+1)}_{m,n}\right)^{\frac{1}{2(n-1)}}.$$

Outline

Freud and Freud-Sobolev type orthogonal polynomials

- Let $\{F_n(x)\}_{n\geqslant 0}$ be Freud polynomials, associated to

$$\langle p,q\rangle = \int_{\mathbb{R}} p(x)q(x)e^{-x^4}dx.$$

- TTRR

$$xF_n(x) = F_{n+1}(x) + a_n^2 F_{n-1}(x), \quad n \geq 1,$$

where the recurrence coefficients a_n satisfy the so called *string equation*

$$4a_n^2\left(a_{n+1}^2 + a_n^2 + a_{n-1}^2\right) = n.$$

- Next, we consider the electrostatic behavior of zeros of the SMOP $\{Q_n(x)\}_{n\geqslant 0}$ corresponding to

$$\langle p,q\rangle_1 = \langle p,q\rangle + M_0 p(0)q(0) + M_1 p'(0)q'(0).$$

- They are known in the literature as Freud-Sobolev type orthogonal polynomials

Electrostatic model for zeros of Q_{2n+1} (odd case)

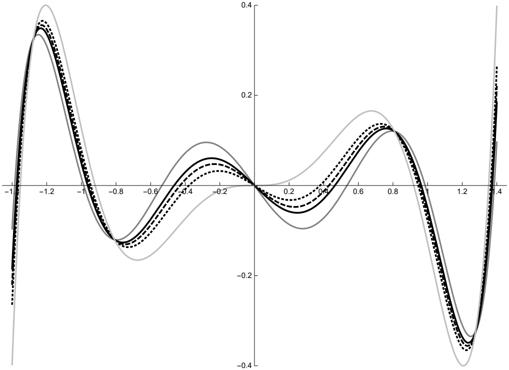
- Here M_0 and M_1 affects independently to the subsequences $\{Q_{2n}(x)\}_{n\geqslant 0}$ and $\{Q_{2n+1}(x)\}_{n\geqslant 0}$ respectively.
- M_1 changes **exclusively** the equilibrium position of the zeros of $\{Q_{2n+1}(x)\}_{n\geqslant 0}$ under the action of the following external potential (Garza, Huertas,Marcellán,(2017))

$$V_{ext}(x,2n+1)=\frac{1}{2}\ln u(x,2n+1)-\frac{1}{2}\ln x^2e^{-x^4}$$

- The first term above represents a *short range potential* corresponding to four unit charges located at the zeros of the **quartic polynomial** $u(x,n)$.
- The coefficients of $u(x,n)$ depend on n, a_n, M_0, M_1 and the confluent Kernels $K_n(0,0)$ and $K_n^{(1,1)}(0,0)$.
- When $u(x,n)\to u(x,2n+1)$ the dependence with M_0 and $K_n(0,0)$ vanishes
- The second term represents a *long range potential* associated with the weight function e^{-x^4}

Electrostatic model for zeros of Q_{2n+1} (odd case)

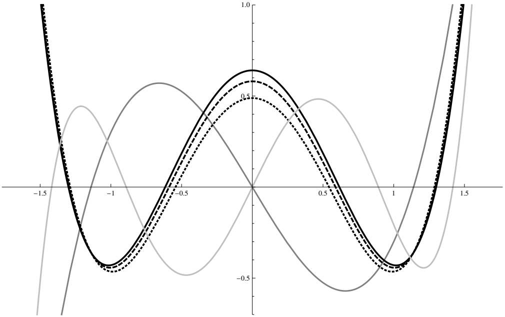
- This picture illustrates the variation of the zeros of the odd degree Freud-Sobolev type polynomials $Q_7(x)$ when M_1 varies.
- Here, the value of M_0 is not relevant.



Electrostatic model for zeros of Q_{2n} (even case)

- The even case was analyzed in (Arceo, Huertas, Marcellán (2016)).
- In a similar way, M_0 changes **exclusively** the equilibrium position of the zeros of the "even" subsequence $\{Q_{2n}(x)\}_{n\geqslant 0}$ under the action of the external potential

$$V_{ext}(x,2n) = \frac{1}{2}\ln u(x,2n) - \frac{1}{2}\ln x^2 e^{-x^4}.$$



Thank you!