Universality for conditional measures of the sine process

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Sine point process

Determinantal point process with correlation kernel

$$\frac{\sin \pi(x-y)}{\pi(x-y)}$$

Figure: **Points from the sine point process**

Figure: **Points from Poisson point process**

Orthogonal Polynomial ensembles

Sine point process is the limit of many OP ensembles, in particular GUE

- **OP ensemble**

$$\frac{1}{Z_n} \prod_{j<k} (x_k - x_j)^2 \prod_{j=1}^{N} w(x_j)$$

- **Orthonormal polynomials** $\varphi_j(x; w)$**,** $j = 0, 1, \ldots$**, give**

$$\widehat{K}_N(x, y; w) = \sum_{j=0}^{N-1} \varphi_j(x; w) \varphi_j(y; w)$$

$$K_N(x, y; w) = \sqrt{w(x)} \sqrt{w(y)}\, \widehat{K}_N(x, y; w)$$

- **OP ensemble is determinantal with correlation kernel** $K_N(x, y; w)$**.**

Universality

After proper rescaling, the OP kernel converges to the sine kernel (in many cases).

- **For varying weights** $w(x) = e^{-NV(x)}$

$$\frac{1}{N\psi_V(x^*)} K_N\left(x^* + \frac{x}{N\psi_V(x^*)}, x^* + \frac{y}{N\psi_V(x^*)}; e^{-NV}\right) \to \frac{\sin \pi(x-y)}{\pi(x-y)} \qquad \text{as } N \to \infty$$

- $\psi_V(x^*)$ **is the density of the equilibrium measure in the external field** V **at the point** x^***.**
- **Universality limit holds if** $\psi_V(x^*) > 0$**.**

Rigidity

The sine point process is rigid

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- **Suppose we observe the points outside a compact interval.**
- **With probability one we know the number of points inside that interval.**

Ghosh (2014), Ghosh and Peres (to appear)

Conditional measure

The conditional measure is the joint distribution of the points inside the interval.

- **Configuration X from sine point process.**
- **Suppose N points inside the interval I.**

- **Bufetov (2016) showed that the conditional measure is OP ensemble with weight**

$$\rho_{I,X}(x) = \prod_{p \in X \setminus I} \left(1 - \frac{x}{p}\right)^2, \qquad x \in I.$$

Bufetov's problem

Alexander Bufetov posed the following problem:

Problem

Let $(p_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers with frequency one and balanced. For $R > 0$ let

$$N = \#\{p_n \mid |p_n| \leq R\}.$$

Consider on $[-R, R]$ the Nth OP ensemble with weight

$$\rho_R(x) = \prod_{n:|p_n|\geq R} \left(1 - \frac{x}{p_n}\right)^2, \qquad x \in [-R, R].$$

Then $\lim_{R\to\infty} K_N(x, y; \rho_R) = \frac{\sin \pi(x-y)}{\pi(x-y)}$ **uniformly on bounded intervals.**

Conditions

- **Frequency one:**

$$\lim_{n\to\pm\infty} \frac{p_n}{n} = 1$$

- **Balanced:**

$$\sum_{n:p_n\neq 0} \frac{1}{p_n}$$

converges in principal value.

Our result

Theorem (K + Mina-Diaz)

Let $(p_n)_{n\in\mathbb{Z}}$ **be a sequence of real numbers with**

$$\lim_{n\to\pm\infty}\frac{p_n}{n}=1,\qquad \sum_{n:p_n\neq 0}\frac{1}{p_n}\quad \textbf{converges in principal value.}$$

For $R>0$**, let**
$$\rho_R(x)=\prod_{n:|p_n|\geq R}\left(1-\frac{x}{p_n}\right)^2.$$

Then as $N,R\to\infty$ **with** $N/R\to 2$**,**

$$\lim_{\substack{N,R\to\infty\\ N/R\to 2}} K_N(x,y;\rho_R)=\frac{\sin\pi(x-y)}{\pi(x-y)}$$

uniformly on bounded intervals.

Heuristics

We work with

$$w_R(x) = \rho_R(Rx) = \prod_{|p_n| \geq R} \left(1 - \frac{Rx}{p_n}\right)^2, \qquad x \in [-1, 1]$$

and we have to prove

$$\lim_{\substack{N,R\to\infty \\ N/R\to 2}} \frac{2}{N} K_N\left(\frac{2x}{N}, \frac{2y}{N}; w_R\right) = \frac{\sin \pi(x-y)}{\pi(x-y)}$$

- **Compare with weights** $w(x) = e^{-NV(x)}$

$$\lim_{N\to\infty} \frac{1}{N\psi_V(0)} K_N\left(\frac{x}{N\psi_V(0)}, \frac{y}{N\psi_V(0)}; e^{-NV}\right) = \frac{\sin \pi(x-y)}{\pi(x-y)}$$

ψ_V **is density of equilibrium measure in external field** V

External field

For large R and N with $N/R \approx 2$,

$$w_R(x) = \prod_{|p_n| \geq R} \left(1 - \frac{Rx}{p_n}\right)^2 \approx e^{-NV(x)}, \qquad x \in [-1, 1]$$

with

$$V(x) = (1+x)\log(1+x) + (1-x)\log(1-x)$$

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- **Equilibrium measure μ_V has constant density**

$$\psi_V \equiv \frac{1}{2}, \qquad \textbf{on } [-1, 1].$$

Lubinsky comparison technique

Lubinsky developed a technique to prove universality by comparing with known weights. It is based on inequalities for

$$\widehat{K}_N(x,y;w) = \sum_{j=0}^{N-1} \varphi_j(x;w)\varphi_j(y;w)$$

- **If** $w_1 \leq w_2$ **then** $\quad \widehat{K}_N(x,x;w_2) \leq \widehat{K}_N(x,x;w_1) \quad$ **and**

$$\left| \widehat{K}_N(x,y;w_1) - \widehat{K}_N(x,y;w_2) \right| \leq \sqrt{\widehat{K}_N(x,x;w_1)\widehat{K}_N(y,y;w_1)} \sqrt{1 - \frac{\widehat{K}_N(x,x;w_2)}{\widehat{K}_N(x,x;w_1)}}$$

Technical estimates on w_R

Proposition

Let $\alpha > 1$ and $0 < \beta < 1$. Then for R large enough (depending on α and β)

$$w_R(x) \leq e^{-N\left(V\left(\frac{x}{\alpha}\right)+\varepsilon_R x\right)}, \qquad x \in [-1,1],$$

and

$$w_R(x) \geq \begin{cases} e^{-N(V(\alpha x)+\varepsilon_R x)}, & x \in [-\beta, \beta], \\ 0, & \textbf{elsewhere,} \end{cases}$$

where the number $\varepsilon_R = \frac{2R}{N} \sum_{|p_n| \geq R} \frac{1}{p_n}$ **is such that the derivatives agree at $x = 0$.**

Note $\varepsilon_R \to 0$ as $R \to \infty$.

Slight modifications

$$w^+_{R,\alpha}(x) = \frac{1}{\sqrt{1-x^2}} e^{-N\left(V\left(\frac{x}{\alpha}\right)+\varepsilon_R x\right)},$$
$$w^-_{R,\alpha}(x) = \sqrt{1-\beta^{-2}x^2}\, e^{-N(V(\alpha x)+\varepsilon_R x)} \chi_{[-\beta,\beta]}(x), \qquad \beta = \alpha^{-2}.$$

Then, for R large enough,

$$w^-_{R,\alpha}(x) \le w_R(x) \le w^+_{R,\alpha}(x)$$

and

$$\lim_{R\to\infty} w^-_{R,\alpha}\left(\frac{x}{R}\right) = \lim_{R\to\infty} w^+_{R,\alpha}\left(\frac{x}{R}\right) = 1$$

Universality for $w^{\pm}_{R,\alpha}$

Proposition

Universality at the origin:

$$\lim_{R\to\infty} \frac{1}{Nc_\alpha^+} K_N\left(\frac{x}{Nc_\alpha^+}, \frac{y}{Nc_\alpha^+}; w^+_{R,\alpha}\right) = \frac{\sin\pi(x-y)}{\pi(x-y)},$$

with constant c_α^+ satisfying $\lim_{\alpha\to 1-} c_\alpha^+ = \frac{1}{2}$.

Similarly for $w^-_{R,\alpha}$.

Universality for $w^{\pm}_{R,\alpha}$

Proposition

Universality at the origin:

$$\lim_{R\to\infty} \frac{1}{Nc^{+}_{\alpha}} K_N\left(\frac{x}{Nc^{+}_{\alpha}}, \frac{y}{Nc^{+}_{\alpha}}; w^{+}_{R,\alpha}\right) = \frac{\sin \pi(x-y)}{\pi(x-y)},$$

with constant c^{+}_{α} satisfying $\lim_{\alpha\to 1-} c^{+}_{\alpha} = \frac{1}{2}$.

Similarly for $w^{-}_{R,\alpha}$.

This is enough to complete the proof of our theorem by playing with the Lubinsky comparison inequalities.

Thank you for your attention !!

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Thanks for the wonderful conference

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When will be the next one?