Concentration of measure for Coulomb gases

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Optimal Point Configurations and Orthogonal Polynomials
CIEM Castro Urdiales - 21 april 2017

Coulomb interaction in $\mathbb{R}^d$

$$g(x-y) := \begin{cases} \log \dfrac{1}{|x-y|} & \text{if } d = 2, \\ \\ \dfrac{1}{|x-y|^{d-2}} & \text{if } d \geq 3. \end{cases}$$

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Key: g is the fundamental solution of Poisson's equation,

$$\Delta g = -c_d \, \delta_0$$

where $c_d := \min(d-2, 1)|\mathbb{S}^{d-1}|$.

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Goal: Study the behavior of the empirical measure

$$\hat{\mu}_N := \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j}$$

When $d = 2$, $\mathbb{R}^d \simeq \mathbb{C}$, the probability distribution reads

$$\mathrm{d}\mathbb{P}(x_1, \dots, x_N) = \frac{1}{Z_N} \prod_{i \neq j} |x_i - x_j|^\beta \prod_{j=1}^{N} \mathrm{e}^{-\beta N V(x_j)} \mathrm{d}x_j.$$

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- If $\beta \to 0$, $x_1,\dots,x_N \sim$ independent random variables
- If $\beta = 1$, $x_1,\dots,x_N \sim$ determinantal point process with

$$K_N(x,y) = \sum_{k=0}^{N-1} P_k(x)\overline{P_k(y)}, \quad \int_{\mathbb{C}} P_k(x)\overline{P_\ell(x)}\mathrm{e}^{-NV(x)}\mathrm{d}x = \delta_{k\ell},$$

P_k polynomials of degree k. Moreover,

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- If $\beta \to \infty$, $x_1, \ldots, x_N$ are minimizers of H_N (Fekete points)

Equilibrium measure: The energy functional

$$\mathscr{E}_V(\mu) := \iint g(x-y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) + \int V(x)\,\mu(\mathrm{d}x)$$

has a unique minimizer μ_{eq}^V over probability measures $\mathcal{P}(\mathbb{R}^d)$.

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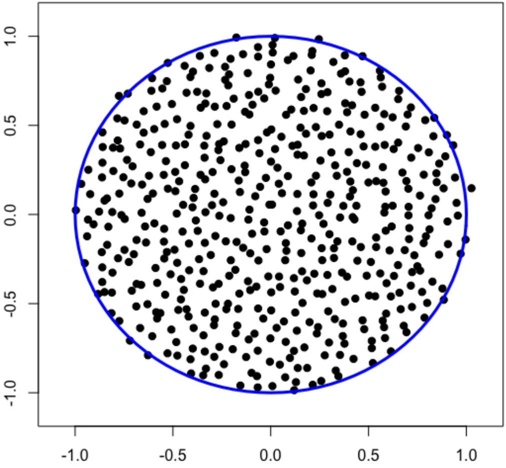
Theorem: With probability one, we have the weak convergence

$$\hat{\mu}_N \xrightarrow[N\to\infty]{} \mu_{\mathrm{eq}}^V$$

$$N = 500, \qquad \beta = 1, \qquad V(x) = |x|^2 \qquad \text{(Ginibre)}$$



How to control the distance from $\hat{\mu}_N$ to μ_{eq}^V?

- Bounded-Lipschitz metric ($\sim$ weak topology)

$$d_{BL}(\mu,\nu) := \sup_{\substack{\|f\|_\infty \le 1 \\ \|\nabla f\|_\infty \le 1}} \left| \int f \, \mathrm{d}(\mu - \nu) \right|$$

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- Large deviation rate function

$$\Phi_V(\mu) := \mathscr{E}_V(\mu) - \mathscr{E}_V(\mu_{\mathrm{eq}}^V)$$

Large deviation principle: For any Borel set $\mathscr{B} \subset \mathcal{P}(\mathbb{R}^d)$,

$$\mathbb{P}(\hat{\mu}_N \in \mathscr{B}) \simeq \exp\{-\beta N^2 \inf_{\mathscr{B}} \Phi_V\}$$

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More precisely,

Theorem: (Chafaï, Golzan, Zitt)

$$\limsup_{N\to\infty} \frac{1}{\beta N^2} \log \mathbb{P}(\hat{\mu}_N \in \mathscr{B}) \leq - \inf_{\text{Closure}(\mathscr{B})} \Phi_V$$

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Corollary: For any fixed $r, \beta > 0$, there exists $c, C, N_0 > 0$,

$$\mathrm{e}^{-cN^2} \leq \mathbb{P}\Big(d_{BL}(\hat{\mu}_N, \mu_{\mathrm{eq}}^V) \geq r\Big) \leq \mathrm{e}^{-CN^2}, \qquad N \geq N_0.$$

Theorem: (Chafaï, H, Maïda) Assume V is $\mathscr{C}^2$ and

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Then there exists $\boldsymbol{A} > 0$, $\boldsymbol{B} \in \mathbb{R}$ and $\boldsymbol{C}(\beta)$ such that:

For any $r, \beta > 0$ and $N \geq 2$,

$$\mathbb{P}\Big(d_{BL}(\hat{\mu}_N, \mu_{\text{eq}}^V) \geq r\Big) \leq \mathrm{e}^{-\boldsymbol{A}\beta N^2 r^2 + \frac{\beta}{4} N \log N\, \mathbf{1}_{d=2} + \boldsymbol{B}\beta N^{2-2/d} + \boldsymbol{C}(\beta) N}.$$

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If further,

- $V(x) \gtrsim c|x|^\kappa$ for some $c, \kappa > 0$, then

$$\boldsymbol{C}(\beta) \sim \begin{cases} \beta & \text{as } \beta \to \infty, \\ \log 1/\beta & \text{as } \beta \to 0. \end{cases}$$

- $V(x) \gtrsim c|x|^2$ for some $c > 0$, then it also holds in W_1.

Corollary: For $\beta > 0$ fixed, there exists $r_0, C > 0$ such that,

$$\mathbb{P}\Big(d_{BL}(\hat{\mu}_N, \mu_{\mathrm{eq}}^V) \geq r\Big) \leq \mathrm{e}^{-CN^2r^2}$$

for any $N \geq 2$ and

$$r \geq r_0 \begin{cases} N^{-1/2}\sqrt{\log N} & \text{if } d = 2 \\ N^{-1/d} & \text{if } d \geq 3 \end{cases}$$

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Corollary: For any $0 \leq s < 1/d$ and $x_0 \in \mathbb{R}^d$,

$$\sup_{\substack{\|f\|_\infty \leq 1 \\ \|\nabla f\|_\infty \leq 1}} \left| \int f\big(N^s(x - x_0)\big) \mathrm{d}(\hat{\mu}_N - \mu_{\mathrm{eq}}^V)(x) \right| \xrightarrow[N\to\infty]{} 0$$

Key of the proof: Coulomb transport inequalities

Theorem: Let $K \subset \mathbb{R}^d$ be compact. There exists $C_K > 0$ such that, for any $\mu, \nu \in \mathcal{P}(K)$ with finite Coulomb energy,

$$W_1(\mu, \nu)^2 \leq C_K \, \mathscr{E}_0(\mu - \nu)$$

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Theorem: If V satisfies

$$V \gg 4 \log |x| \mathbf{1}_{d=2},$$

then exists $C_{BL}^V > 0$ such that, for any $\mu \in \mathcal{P}(\mathbb{R}^d)$,

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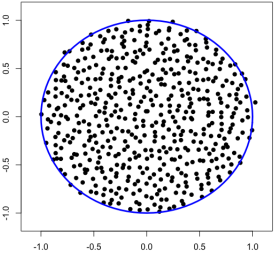
In the Ginibre setting: $d = 2,\ \beta = 1,\ V(x) = |x|^2$.

$$\mu_{\text{eq}}(\mathrm{d}x) = \frac{\mathbf{1}_{\mathbb{D}}}{\pi}\,\mathrm{d}x$$



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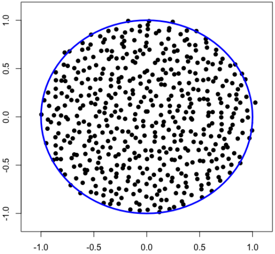
Setting $\boldsymbol{\xi} := 1/C_{W_1}^{|\cdot|^2}$, for any $N \geq 2$ and $r > 0$,

$$\mathbb{P}\Big(W_1(\hat{\mu}_N, \mu_{\mathrm{eq}}) \geq r\Big) \leq \mathrm{e}^{-\frac{1}{4}\boldsymbol{\xi} N^2 r^2 + \frac{1}{2} N \log N + [\boldsymbol{\xi} + \frac{3}{2} - \log \pi] N}.$$



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