Dynamics of a planar Coulomb gas

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Outline

Diffusions

- Markov process $(X_t)_{t\geqslant 0}$ Stochastic Differential Equation

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- Reversible (and thus invariant) Boltzmann-Gibbs measure

$$\mu(\mathrm{d}x) = \frac{\mathrm{e}^{-H(x)}}{Z}\mathrm{d}x$$

$$X_0 \sim \mu \quad \Rightarrow \quad (X_0, X_t) \overset{d}{=} (X_t, X_0) \quad \forall t \geqslant 0$$

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$$\mathrm{d}X_t = \sqrt{2\alpha}\mathrm{d}B_t - \alpha\nabla H(X_t)\mathrm{d}t$$

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$$\mathrm{d}X_t = \sqrt{2\frac{\alpha}{\beta}}\mathrm{d}B_t - \alpha\nabla H(X_t)\mathrm{d}t$$

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$$\mu_\beta(\mathrm{d}x) = \frac{\mathrm{e}^{-\beta H(x)}}{Z_\beta}\mathrm{d}x$$

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- Infinitesimal generator $G = \Delta - \nabla H \cdot \nabla$

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- The operators G and $S_t = \mathrm{e}^{tG}$ are symmetric in $L^2(\mu)$
- Fokker-Planck equation if $f_t = \frac{\mathrm{d}\mu_t}{\mathrm{d}\mu}$ with $\mu_t = \mathrm{Law}(X_t)$ then

$$f_t = S_t(f_0) \quad \text{and} \quad \partial_t f_t = G f_t$$

Exactly solvable model: Ornstein-Uhlenbeck process

- Gaussian model: $H(x) = \frac{1}{2}|x|^2$, $\mathrm{d}X_t = \sqrt{2}\mathrm{d}B_t - X_t\mathrm{d}t$,

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- Poincaré inequality:
$$\mathrm{Var}_\mu(f) \leqslant -\mathbb{E}_\mu(fGf) = \mathbb{E}_\mu(|\nabla f|^2)$$

Comparison to Gaussianity via convexity

Theorem (Brascamp–Lieb 1976)

If $\nabla^2 H > 0$ *on* $\mathbb{R}^d$ *then for any smooth* $f : \mathbb{R}^d \to \mathbb{R}$*,*

$$\mathrm{Var}_\mu(f) \leqslant \mathbb{E}_\mu \langle (\nabla^2 H)^{-1} \nabla f, \nabla f \rangle$$

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- Jensen divergence: $\mathrm{Var}_\mu(f) = \mathbb{E}_\mu \Phi(f) - \Phi(\mathbb{E}_\mu f)$, $\Phi(u) = u^2$

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Theorem (Bakry–Émery 1984)

If $\nabla^2 H \geqslant \rho > 0$ *then for any convex* $\Phi : I \to \mathbb{R}$ *with* $(u,v) \mapsto \Phi''(u)v^2$ *convex and any smooth* $f : \mathbb{R}^d \to I$

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- Logarithmic Sobolev: $I = \mathbb{R}_+$, $\Phi(u) = u \log(u)$

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Theorem (Caffarelli 2000)

If $\nabla^2 H \geqslant \rho > 0$ *then* μ *is the image of* $\mathcal{N}(0, I_d)$ *by a Lipschitz function* $F : \mathbb{R}^d \to \mathbb{R}^d$ *with* $\|F\|_{\mathrm{Lip}} \leqslant \frac{1}{\sqrt{\rho}}$.

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\begin{aligned}
\mathrm{Var}_\mu(f) &= \mathrm{Var}_{\mathcal{N}(0,I_d)}(f(F)) \\
&\leqslant \mathbb{E}_{\mathcal{N}(0,I_d)}(|\nabla f(F)|^2) \\
&\leqslant \frac{1}{\rho}\mathbb{E}_\mu(|\nabla f|^2)
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- Gives also any Φ-Sobolev inequality from the Gaussian!

KLS conjecture

Conjecture (Kannan–Lovász–Simonovits 1995)

If $\nabla^2 H \geqslant 0$ with $\mathbb{E}_\mu(x_i x_j) = \delta_{ij}$ then μ satisfies to a Poincaré inequality with a universal constant (independent of d and H).

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- Lee & Vempala 2016: true with $d^{1/4}$

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- Stochastic Differential Equation à la Ornstein–Uhlenbeck

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- Erdős–Yau: book *Dynamical approach to RMT*

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Outline

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Theorem (Second moment dynamics)

For any $x \in D$ *and* $t \geqslant 0$, *we have*

$$\mathbb{E}\left(\frac{|X_t|^2}{N} \mid X_0 = x\right) = \frac{|x|}{N}\mathrm{e}^{-\frac{4\alpha_N}{N}t} + \left(\frac{1}{2} + \frac{N}{\beta_N} - \frac{1}{2N}\right)\left(1 - \mathrm{e}^{-\frac{4\alpha_N}{N}t}\right).$$

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- Second moment of φ bounded in N (then Bobkov theorem)

McKean–Vlasov mean-field limit

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Theorem (Limiting McKean–Vlasov equation)

If $\sigma = \lim_{N\to\infty} \frac{\alpha_N}{\beta_N} \in [0,\infty)$ *then* $\lim_{N\to\infty} \nu_{N,t} = \nu_t$ *with*

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or more compactly: $\partial_t\nu_t = \sigma\Delta\nu_t + \nabla\cdot((\nabla V + \nabla W * \nu_t)\nu_t)$.

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- Semi-linear Partial Differential Equation (Sznitman)
- Regimes: $(\alpha_N, \beta_N) = (N, N^2)$ and $(\alpha_N, \beta_N) = (N, N)$

MIND THE GAP

That's all folks!

Thank you for your attention.