Matrix Orthogonal Polynomials and Time and Band limiting

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Outline

1. Introduction
   - The origins of time and band limiting
   - Setting of the problem in the matrix case
2. Matrix Orthogonal Polynomials and band and time limiting
   - The *discrete version* of the problem
   - The *continuous version* of the problem, an example

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The origins of time and band limiting

C. Shannon, *A mathematical theory of communication*, Bell Tech. J., vol **27**, 1948, pp 379–423 (July) and pp 623–656 (Oct).

Shannon's Problem:

Consider an unknow signal $f(t)$ of finite duration, i.e. with support in $[-T, T]$ (time limiting); the data are the values of its Fourier transform $F(k)$ for k in the interval $[-\mathcal{W}, \mathcal{W}]$. (band limiting)

In practice one only has noisy values of $F(k)$.

What is the best use of this information?

Though this problem was posed originally by Shannon, a full solution can be found in joint works by: David Slepian, Henry Landau and Henry Pollak. (Bell labs,1960's)

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The origins of time and band limiting

In certain areas of Mathematics one arrives at a global operator, given by an integral kernel or a full matrix. One needs to compute numerically many of its eigenfunctions in an efficient way.

In certain cases one can exhibit a differential operator of low order, or a narrow band matrix, which has the same eigenfuntions as the global operator. The numerical computation of the eigenfunctions of the differential operator, a local one, is *much easier* than the initial task.

The origins of time and band limiting

Shannon- Slepian- Landau- Pollak :

$$A = [-T, T], \qquad B = [-\mathcal{W}, \mathcal{W}],$$

$$(\mathscr{I} f)(x) = \int_{-T}^{T} \frac{\sin(\mathcal{W}(x-y))}{x-y} f(y)\, dy, \qquad x \in A.$$

The operator

$$(Df)(x) = ((T^2 - x^2) f'(x))' - \mathcal{W}^2 x^2 f$$

has simple spectrum and an appropriate selfadjoint extension of D commutes with $\mathscr{I}$.

The origins of time and band limiting

The *kernel* is obtained by integrating e^{isx} in $[-\mathcal{W}, \mathcal{W}]$ and one has an integral operator acting on functions defined in $[-T, T]$.

$$k(x,y) = \frac{\sin(\mathcal{W}(x-y))}{x-y} = \int_{-\mathcal{W}}^{\mathcal{W}} e^{isx} e^{-isy} ds$$

The origins of time and band limiting

How to find (and explain the existence of...) the differential operator D that will commute with an integral operator $\mathscr{I}$ with kernel $k(x,y)$ acting on $L^2[a,b]$?

Some useful guide-references

- F. A. Grünbaum, *Eigenvectors of a Toeplitz matrix: discrete version of the prolate spheroidal wave functions*, SIAM J. on Algebraic Discrete Methods **2** (1981), 136–141.
- F. A. Grünbaum, *A new property of reproducing kernels of classical orthogonal polynomials*, J. Math. Anal. Applic. **95** (1983), 491–500.
- F. A. Grünbaum, L. Longhi, and M. Perlstadt, *Differential operators commuting with finite convolution integral operators: some nonabelian examples*, SIAM J. Appl. Math. **42** (1982), 941–955.

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The matrix case

In this talk we will consider an example of matrix valued orthogonal polynomials satisfying differential equations (i.e a bispectral situation) in connection with time and band limiting.

- **"discrete version" of time and band limiting:** one deals with a global operator given by a full matrix and one looks for a commuting local object given by a tridiagonal matrix.
- **"continuous version" of time and band limiting:** In this talk we deal with a global operator given by an integral operator and one looks for a commuting local object given by a second order differential operator

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Matrix orthogonal polynomials

We consider a sequence of matrix orthonormal polynomials Q_n w.r.t a matrix weight $W(t)$, of dimension R, supported in an interval $[a, b]$:

$$\langle Q_i, Q_j \rangle_W = \int_a^b Q_i(x) W(x) Q_j(x)^* dx = \delta_{ij} \mathcal{I}$$

- Q_n are matrix polynomials with non singular leading coefficients

$$Q_n(x) = A_n x^n + A_{n-1} x^{n-1} + \ldots + A_0, \quad A_n \in M_R(\mathbb{R}).$$

- The sequence Q_n is unique up to the multiplication on the left by a unitary matrix.

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The noncommutative setting

Consider the following two Hilbert spaces:

- The space $L^2(W)$, of all measurable matrix valued functions $f(x)$, $x \in (a,b)$, satisfying $\int_a^b \mathrm{tr}\,(f(x)W(x)f^*(x))\,dx < \infty$.
- The space $\ell^2(M_R, \mathbb{N}_0)$ of all real valued $R \times R$ matrix sequences $(C_w)_{w\in\mathbb{N}_0}$ such that $\sum_{w=0}^{\infty} \mathrm{tr}\,(C_w C_w^*) < \infty$.

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The noncommutative setting

The map $\mathcal{F} : \ell^2(M_R, \mathbb{N}_0) \longrightarrow L^2(W)$ (analogy to the Fourier Transform) given by

$$(A_w)_{w=0}^{\infty} \longmapsto \sum_{w=0}^{\infty} A_w Q_w(x)$$

is an isometry. If the polynomials are dense in $L^2(W)$, this map is unitary with the inverse $\mathcal{F}^{-1} : L^2(W) \longrightarrow \ell^2(M_R, \mathbb{N}_0)$ given by

$$f \longmapsto A_w = \int_a^b f(x)\, W(x)\, Q_w^*(x) dx.$$

- Here $\mathbb{N}_0$ takes up the role of "physical space"
- The interval (a, b) the role of "frequency space".

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Consider the problem of determining a function f, from the following data:

- f has support on the finite set $\{0, \ldots, N\}$
- its Fourier transform $\mathcal{F}f$ is known on the compact set $[a, \Omega]$.

This can be formalized as follows

$$\chi_\Omega \mathcal{F} f = g = \text{ known }, \qquad \chi_N f = f.$$

We can combine the two equations into

$$Ef = \chi_\Omega \mathcal{F} \chi_N f = g.$$

To analyze this problem we need to compute the singular vectors (and values) of the operator $E : \ell^2(M_R, \mathbb{N}_0) \longrightarrow L^2(W)$. These are given by the eigenvectors of the operators

$$E^*E = \chi_N \mathcal{F}^{-1} \chi_\Omega \mathcal{F} \chi_N \qquad \text{and} \qquad EE^* = \chi_\Omega \mathcal{F} \chi_N \mathcal{F}^{-1} \chi_\Omega.$$

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The global operators

Consider now the problem of finding the eigenfunctions of E^*E and EE^*. For arbitrary N and Ω there is no hope of doing this analytically, and one has to resort to numerical methods and this is not an easy problem.

- The operator E^*E, acting in $\ell^2(M_R, \mathbb{N}_0)$ is just a finite dimensional block-matrix M (discrete), and each block is given by

$$(M)_{m,n} = (E^*E)_{m,n} = \int_a^\Omega Q_m(x)W(x){Q^*}_n(x)dx, \quad 0 \leq m, n \leq N$$

The second operator $S = EE^*$ acts in $L^2((a,\Omega), W(t)dt)$ by means of the integral kernel (continuous)

$$k(x,y) = \sum_{n=0}^{N} Q_n^*(x)Q_n(y).$$

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Q_n matrix orthonormal polynomials w.r.t a matrix weight $W(t)$ supported for instance in the interval $[-1,1]$.

One fixes a natural **even** number N and $\Omega \in (-1,1]$ and consider the matrix M of total size $N \times N$,

$$E^*E = M = \begin{pmatrix} M^{0,0} & M^{0,1} & \dots & M^{0,\frac{N}{2}-1} \\ M^{1,0} & M^{1,1} & \dots & M^{1,\frac{N}{2}-1} \\ \dots & \dots & \dots & \dots \\ M^{\frac{N}{2}-1,0} & M^{\frac{N}{2}-1,1} & \dots & M^{\frac{N}{2}-1,\frac{N}{2}-1} \end{pmatrix},$$

$$M^{i,j} = \int_{-1}^{\Omega} Q_i(x)W(x)Q_j(x)^* dx, \quad \text{for } 0 \le i,j \le \frac{N}{2} - 1$$

- the restriction to the interval $[-1,\Omega]$ implements "band-limiting"
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Searching for a commuting block-tridiagonal matrix L

The problem is to find all block tridiagonal symmetric matrices L such that

$$ML = LM.$$

$$L = \begin{pmatrix} L^{1,1} & L^{1,2} & 0 & \cdots & & 0 \\ L^{2,1} & L^{2,2} & L^{2,3} & 0 & \cdots & \\ \cdots & \ddots & \ddots & \ddots & & \\ & \cdots & & L^{\frac{N}{2}-1,\frac{N}{2}-2} & L^{\frac{N}{2}-1,\frac{N}{2}-1} & L^{\frac{N}{2},\frac{N}{2}} \\ 0 & \cdots & & & L^{\frac{N}{2},\frac{N}{2}} & 0 \end{pmatrix}.$$

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- Notice that in principle there is not guarantee that one my find any such L except for a scalar multiple of the identity.
- We need matrices L that have a simple spectrum.
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The analog with Shanon's problem

time limiting $T \longrightarrow N$ size of the matrix M

band limiting $\mathcal{W} \longrightarrow \Omega$ upper bound in the definition of $M^{i,j}$

integral operator $\mathscr{I} \longrightarrow M$ matrix of truncated inner products

differential operator $D \longrightarrow L$ matrix with a "narrow" band

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The first examples of band and time limiting in the matrix case as a natural extension of previous works of A. Grünbaum and A. Grünbaum-L. Longhi-M. Perlstadt can be found in:

- F. A. Grünbaum, I. Pacharoni and I. Zurrian, *Time and band limiting for matrix valued functions,an Example*, SIGMA 11 (2015), 044.
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Continuous version" of time and band limiting

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- M. Castro, F.A. Grünbaum, I. Pacharoni and I. Zurrian, *A further look at time and band limiting for matrix orthogonal polynomials*, (2017), arXiv:1703.06942. To appear in "*Frontiers in Orthogonal Polynomials and q-Series*", World Scientific.

"Continuous version": when the global operator is given by an integral operator, a Jacobi type example

For $\alpha, \beta > -1$, the scalar Jacobi weight is given by

$$w_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}, \qquad (1)$$

supported in the interval $[-1, 1]$.

We consider a Jacobi type weight matrix of dimension two [CGPZ 2017]

$$W(x) = W_{(\alpha,\beta)} = \frac{1}{2}\begin{pmatrix} w_{\alpha,\beta} + w_{\beta,\alpha} & -w_{\alpha,\beta} + w_{\beta,\alpha} \\ -w_{\alpha,\beta} + w_{\beta,\alpha} & w_{\alpha,\beta} + w_{\beta,\alpha} \end{pmatrix}, \quad x \in [-1, 1].$$

A particular case of these weight matrices has already been considered in previous work by Grunbaum-Pacharoni-Zurrián (2015, 2017) and Castro-Grunbaum (2015).

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where $p_n^{(\alpha,\beta)}$ are the classical Jacobi polynomials

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Our polynomials P_n are eigenfunctions of the second order differential operator

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We have that the differential operator D can be factorized as

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Therefore the sequence of matrix orthogonal polynomials $(P_n(x))_n$ satisfies

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The integral kernel

One considers the Integral kernel

$$k_N(s,t) = \sum_{n=0}^{N} Q_n^*(t) Q_n(s).$$

where $Q_n(x) = h_n^{-1/2} P_n(x)$ is the sequence of orthonormal polynomials.

Here $h_n \mathrm{Id} = ||P_n(x)||^2$,

$$h_n = \frac{2^{\alpha+\beta+1}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{(\alpha+\beta+2n+1)\,n!\,\Gamma(\alpha+\beta+n+1)}.$$

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$$I_k(F) = \int_{-1}^{\Omega} F(s)W(s)(k_N(s,t))^* ds$$

We search for an operator

$$\widetilde{D} = \frac{d^2}{dt^2}\widetilde{E}_2(t) + \frac{d}{dt}\widetilde{E}_1(t) + \frac{d^0}{dt}\widetilde{E}_0(t)$$

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Observe that we have two parameters Ω and N:

$$[-1, \Omega] : \qquad \text{“band-limiting”}$$

$$0, 1, \ldots, N \qquad \text{“time-limiting”}$$

The commuting differential operator

Theorem

Let $p(x) = (1 - x^2)W(x)$. *The symmetric second-order differential operator*

$$\widetilde{D} = \frac{d}{dx}\left((x - \Omega)\frac{d}{dx}p(x)\right)W(x)^{-1} + x\,A, \qquad (2)$$

with $A = N(N + \alpha + \beta + 2)\mathrm{Id}$, *commutes with the integral operator* I_k *given by the integral kernel* $k_N(s,t)$

The vector space of such operators has dimension two except for trivial commuting operators of order zero.

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The operator $\widetilde{D}$ is symmetric with respect to the weight matrix W, i., e.,

$$\langle f\widetilde{D}, g\rangle_\Omega = \int_{-1}^{\Omega} f(x)\widetilde{D}W(x)g^*(x)\,dx = \langle f, g\widetilde{D}\rangle_\Omega$$

The commuting differential operator

Explicitly, we have

$$\widetilde{D} = \frac{d^2}{dx^2}\widetilde{E}_2(x) + \frac{d}{dx}\widetilde{E}_1(x) + \widetilde{E}_0(x) \tag{4}$$

where the coefficients $\widetilde{E}_j,\ j = 0, 1, 2$, are given by

$$\widetilde{E}_2 = (x - \Omega)(1 - x^2)\mathrm{Id},$$
$$\widetilde{E}_1 = \Big(-(3 + \alpha + \beta)x^2 + x\,\Omega(2 + \alpha + \beta) + 1\Big)\mathrm{Id} + (\alpha - \beta)(x - \Omega)T,$$
$$\widetilde{E}_0 = x\,N(N + \alpha + \beta + 2)\mathrm{Id},$$

and T is the permutation matrix

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The commuting differential operator

The sequence of orthonormal polynomials satisfy a second orden differential equation

$$Q_n(x)D = \Upsilon_n Q_n(x),$$

for certain scalar eigenvalue sequence Υ_n and

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Main tools in the proof

- Differentiation formula

$$(1-x^2)\frac{d}{dx}P_n(x) = -nxP_n(x) - \frac{n(\alpha-\beta)}{\alpha+\beta+2n}T\,P_n(x) + \gamma_{n-1}P_{n-1}(x),$$

where

$$\gamma_{n-1} = \frac{2(n+\alpha)(n+\beta)}{\alpha+\beta+2n}.$$

Main tools in the proof

- Christoffel Darboux formula

$$\frac{\kappa_{n-1}}{\kappa_n\, h_{n-1}}\left(P_{n-1}^*(y)P_n(x) - P_n^*(y)P_{n-1}(x)\right) = (x-y)\sum_{k=0}^{n-1}\frac{P_k^*(y)P_k(x)}{h_k}$$

with

$$\kappa_n = \frac{\Gamma(\alpha+\beta+2n+1)}{2^n\, n!\,\Gamma(\alpha+\beta+n+1)}, \qquad h_n = ||P_n(x)||^2$$

Observe that $\kappa_n \mathrm{Id}$ is the leading coefficient of the matrix polynomial $P_n(x)$ and we also have

$$\frac{\kappa_{n-1}}{\kappa_n} = \frac{2n(n+\alpha+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta-1)}.$$

A Chebyshev type example

As a particular case one may consider $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$ in

$$W_{(\alpha,\beta)} = \frac{1}{2}\begin{pmatrix} w_{\alpha,\beta} + w_{\beta,\alpha} & -w_{\alpha,\beta} + w_{\beta,\alpha} \\ -w_{\alpha,\beta} + w_{\beta,\alpha} & w_{\alpha,\beta} + w_{\beta,\alpha} \end{pmatrix}, \quad x \in [-1,1].$$

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$$W_{(\frac{1}{2},-\frac{1}{2})}(x) = \frac{1}{\sqrt{1-x^2}}\begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}, \quad x \in [-1,1],$$

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The monic family of polynomials orthogonal with respect to this weight matrix, is given explicitly in terms of the Chebyshev polynomials of the second kind $U_n(x)$,

$$\widetilde{P}_n(x) = \frac{1}{2^n}\begin{pmatrix} U_n(x) & -U_{n-1}(x) \\ -U_{n-1}(x) & U_n(x) \end{pmatrix},$$

Moreover, the polynomials $P_n(x)$ orthogonal with respect to the Chebyshev weight $W_{(\frac{1}{2},-\frac{1}{2})}(x)$ satisfy, the first order differential equation

$$P_n'(x)\begin{pmatrix} -x & 1 \\ -1 & -x \end{pmatrix} + P_n(x)\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -n-1 & 0 \\ 0 & n \end{pmatrix} P_n(x),$$

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One considers here the integral operator $I_k$defined by the integral kernel

$$k(x,y) = \sum_{n=0}^{N} 4^n \widetilde{P}_n(x)^* \widetilde{P}_n(y).$$

Particularly, the norm of the polynomials $\widetilde{P}_n$ is given by $||\widetilde{P}_n|| = \frac{\sqrt{\pi}}{2^n}$.

Hence, for this particular example, the commuting operator $\widetilde{D}$ is given by

$$\widetilde{D} = \frac{d^2}{dx^2}(1-x^2)(x-\Omega) + \frac{d}{dx}\Big((-3x^2+2\Omega\, x+1)\mathrm{Id}+(x-\Omega)T\Big)+N(N+2)x$$

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Some final remarks

- When dealing with a global operator given by an integral kernel the main problem is given a matrix weight, and the associated second order differential operator D, to find the commuting operator $\widetilde{D}$ in terms of D.
- One needs to address the issue of the "simplicity of the spectrum" of $\widetilde{D}$ in the matrix valued context in order to guarantee that the eigenfunctions of the integral and the differential operators are the same.
- An interesting problem is to see how time and band survives after a Darboux process, i.e., if one can still guarantee the existence of a commuting local operator, as it was considered for a global operator given by a full matrix by C. and Grunbaum, *Linear algebra and approx.*, (2015).

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