Discrepancy and energy optimization on the sphere

Dmitriy Bilyk
University of Minnesota

"Optimal point configurations and orthogonal polynomials"

CIEM, Castro Urdiales, Cantabria, Spain
April 2017

April 19, 2017

Good point distributions

- Lattices
- Energy minimization
- Monte-Carlo
- Other random point processes (jittered sampling, determinantal point process)
- Covering/packing problems
- Low-discrepancy sets
- Cubature formulas, numerical integration
- Uniform tessellation, almost isometric embeddings

Discrepancy

- U: a set with a natural probability measure μ (e.g., $[0,1]^d$, $\mathbb{S}^d$, $\mathbb{R}^d$, etc.)

Discrepancy

- U: a set with a natural probability measure μ (e.g., $[0,1]^d$, $\mathbb{S}^d$, $\mathbb{R}^d$, etc.)
- $\mathcal{A}$ - a collection of subsets of U ("test sets", e.g., balls, cubes, convex sets, spherical caps)

Discrepancy

- U: a set with a natural probability measure μ (e.g., $[0,1]^d$, $\mathbb{S}^d$, $\mathbb{R}^d$, etc.)
- $\mathcal{A}$ - a collection of subsets of U ("test sets", e.g., balls, cubes, convex sets, spherical caps)
- Choose an N-point set in $Z \subset U$
- Discrepancy of Z with respect to $\mathcal{A}$:

$$D_{\mathcal{A}}(Z) = \sup_{A \in \mathcal{A}} \left| \frac{\#(Z \cap A)}{N} - \mu(A) \right|.$$

Discrepancy

- U: a set with a natural probability measure μ (e.g., $[0,1]^d$, $\mathbb{S}^d$, $\mathbb{R}^d$, etc.)
- $\mathcal{A}$ - a collection of subsets of U ("test sets", e.g., balls, cubes, convex sets, spherical caps)
- Choose an N-point set in $Z \subset U$
- Discrepancy of Z with respect to $\mathcal{A}$:

$$D_{\mathcal{A}}(Z) = \sup_{A \in \mathcal{A}} \left| \frac{\#(Z \cap A)}{N} - \mu(A) \right|.$$

- Optimal discrepancy wrt $\mathcal{A}$:

$$D_N(\mathcal{A}) = \inf_{\#Z = N} D_{\mathcal{A}}(Z).$$

Discrepancy

- U: a set with a natural probability measure μ (e.g., $[0,1]^d$, $\mathbb{S}^d$, $\mathbb{R}^d$, etc.)
- $\mathcal{A}$ - a collection of subsets of U ("test sets", e.g., balls, cubes, convex sets, spherical caps)
- Choose an N-point set in $Z \subset U$
- Discrepancy of Z with respect to $\mathcal{A}$:

$$D_{\mathcal{A}}(Z) = \sup_{A \in \mathcal{A}} \left| \frac{\#(Z \cap A)}{N} - \mu(A) \right|.$$

- Optimal discrepancy wrt $\mathcal{A}$:

$$D_N(\mathcal{A}) = \inf_{\#Z=N} D_{\mathcal{A}}(Z).$$

- sup $\rightarrow L^2$-average: L^2 discrepancy.

Spherical cap discrepancy

For $x \in \mathbb{S}^d$, $t \in [-1,1]$ define spherical caps:

$$C(x,t) = \{y \in \mathbb{S}^d : \langle x, y \rangle \geq t\}.$$

For a finite set $Z = \{z_1, z_2, ..., z_N\} \subset \mathbb{S}^d$ define

$$D_{cap}(Z) = \sup_{x \in \mathbb{S}^d, t \in [-1,1]} \left| \frac{\#\big(Z \cap C(x,t)\big)}{N} - \sigma\big(C(x,t)\big) \right|.$$

Theorem (Beck, '84)

There exists constants c_d, $C_d > 0$ such that

$$c_d N^{-\frac{1}{2}-\frac{1}{2d}} \leq \inf_{\#Z=N} D_{cap}(Z) \leq C_d N^{-\frac{1}{2}-\frac{1}{2d}} \sqrt{\log N}.$$

Spherical caps: L^2 Stolarsky Principle

Define the spherical cap L^2 discrepancy

$$D_{cap,L^2}(Z) = \left(\int_{\mathbb{S}^d} \int_{-1}^{1} \left| \frac{\#\big(Z \cap C(x,t)\big)}{N} - \sigma\big(C(x,t)\big) \right|^2 dt\, d\sigma(x) \right)^{\frac{1}{2}}.$$

Spherical caps: L^2 Stolarsky Principle

Define the spherical cap L^2 discrepancy

$$D_{cap,L^2}(Z) = \left(\int_{\mathbb{S}^d} \int_{-1}^{1} \left| \frac{\#\big(Z \cap C(x,t)\big)}{N} - \sigma\big(C(x,t)\big) \right|^2 dt\, d\sigma(x) \right)^{\frac{1}{2}}.$$

Theorem (Stolarsky invariance principle)

For any finite set $Z = \{z_1, ..., z_N\} \subset \mathbb{S}^d$

$$\frac{1}{N^2} \sum_{i,j=1}^{N} \|z_i - z_j\| + c_d \Big[D_{L^2,cap} \Big]^2 = \text{const}$$

$$= \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \|x - y\|\, d\sigma(x) d\sigma(y).$$

Spherical caps: L^2 Stolarsky Principle

Define the spherical cap L^2 discrepancy

$$D_{cap,L^2}(Z) = \left(\int_{\mathbb{S}^d} \int_{-1}^{1} \left| \frac{\#\big(Z \cap C(x,t)\big)}{N} - \sigma\big(C(x,t)\big) \right|^2 dt\, d\sigma(x) \right)^{\frac{1}{2}}.$$

Theorem (Stolarsky invariance principle)

For any finite set $Z = \{z_1, ..., z_N\} \subset \mathbb{S}^d$

$$c_d \Big[D_{cap,L^2}(Z) \Big]^2 =$$

$$= \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \|x - y\| \, d\sigma(x) d\sigma(y) - \frac{1}{N^2} \sum_{i,j=1}^{N} \|z_i - z_j\|.$$

Spherical caps: L^2 Stolarsky Principle

Define the spherical cap L^2 discrepancy

$$D_{cap,L^2}(Z) = \left(\int_{\mathbb{S}^d} \int_{-1}^{1} \left| \frac{\#\big(Z \cap C(x,t)\big)}{N} - \sigma\big(C(x,t)\big) \right|^2 dt\, d\sigma(x) \right)^{\frac{1}{2}}.$$

Theorem (Stolarsky invariance principle)

For any finite set $Z = \{z_1, ..., z_N\} \subset \mathbb{S}^d$

$$c_d \Big[D_{cap,L^2}(Z) \Big]^2 =$$
$$= \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \|x-y\|\, d\sigma(x) d\sigma(y) - \frac{1}{N^2} \sum_{i,j=1}^{N} \|z_i - z_j\|.$$

- Stolarsky '73, Brauchart, Dick '12, DB, Dai, Matzke '16.

Spherical caps: L^2 Stolarsky Principle

- Define the spherical cap discrepancy of fixed height t:

$$D^{(t)}_{L^2,\mathrm{cap}}(Z) := \left(\int_{\mathbb{S}^d} \left| \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}_{C(x,t)}(z_j) - \sigma\big(C(x,t)\big) \right|^2 d\sigma(x) \right)^{1/2}$$

Spherical caps: L^2 Stolarsky Principle

- Define the spherical cap discrepancy of fixed height t:

$$D^{(t)}_{L^2,\text{cap}}(Z) := \left(\int_{\mathbb{S}^d} \left| \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}_{C(x,t)}(z_j) - \sigma\big(C(x,t)\big) \right|^2 d\sigma(x) \right)^{1/2}$$

-

$$\left[D^{(t)}_{L^2,\text{cap}}(Z) \right]^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} \sigma\big(C(z_i,t) \cap C(z_j,t)\big) - \big(\sigma\big(C(p,t)\big)\big)^2 .$$

Spherical caps: L^2 Stolarsky Principle

- Define the spherical cap discrepancy of fixed height t:

$$D^{(t)}_{L^2,\mathrm{cap}}(Z) := \left(\int_{\mathbb{S}^d} \left| \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}_{C(x,t)}(z_j) - \sigma\big(C(x,t)\big) \right|^2 d\sigma(x) \right)^{1/2}$$

-

$$\left[D^{(t)}_{L^2,\mathrm{cap}}(Z) \right]^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} \sigma\big(C(z_i,t) \cap C(z_j,t)\big) - \big(\sigma\big(C(p,t)\big)\big)^2 .$$

- Averaging over $t \in [-1,1]$

Spherical caps: L^2 Stolarsky Principle

- Define the spherical cap discrepancy of fixed height t:

$$D^{(t)}_{L^2,\mathrm{cap}}(Z) := \left(\int_{\mathbb{S}^d} \left| \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}_{C(x,t)}(z_j) - \sigma\big(C(x,t)\big) \right|^2 d\sigma(x) \right)^{1/2}$$

- $$\left[D^{(t)}_{L^2,\mathrm{cap}}(Z) \right]^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} \sigma\big(C(z_i,t) \cap C(z_j,t)\big) - \big(\sigma\big(C(p,t)\big)\big)^2 .$$

- Averaging over $t \in [-1,1]$

$$\int_{-1}^{1} \sigma\big(C(x,t) \cap C(y,t)\big)\, dt \quad = \quad 1 - C_d \|x-y\|$$

Spherical caps: L^2 Stolarsky Principle

- Define the spherical cap discrepancy of fixed height t:

$$D^{(t)}_{L^2,\mathrm{cap}}(Z) := \left(\int_{\mathbb{S}^d} \left| \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}_{C(x,t)}(z_j) - \sigma\big(C(x,t)\big) \right|^2 d\sigma(x) \right)^{1/2}$$

- $$\left[D^{(t)}_{L^2,\mathrm{cap}}(Z) \right]^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} \sigma\big(C(z_i,t) \cap C(z_j,t)\big) - \big(\sigma\big(C(p,t)\big)\big)^2 .$$

- Averaging over $t \in [-1,1]$

$$\int_{-1}^{1} \sigma\big(C(x,t) \cap C(y,t)\big)\, dt \quad = \quad 1 - C_d \|x-y\|$$

$$\int_{-1}^{1} \big(\sigma\big(C(p,t)\big)\big)^2\, dt \quad = \quad 1 - C_d \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \|x-y\|\, d\sigma(x) d\sigma(y).$$

Hemisphere discrepancy

- L^2 discrepancy for spherical cap discrepancy of fixed height t satisfies:
$$\left[D^{(t)}_{L^2,\mathrm{cap}}(Z)\right]^2 = \frac{1}{N^2}\sum_{i,j=1}^{N} \sigma\big(C(z_i,t)\cap C(z_j,t)\big) - \big(\sigma\big(C(p,t)\big)\big)^2 .$$

Hemisphere discrepancy

- L^2 discrepancy for spherical cap discrepancy of fixed height t satisfies:
$$\left[D^{(t)}_{L^2,\text{cap}}(Z)\right]^2 = \frac{1}{N^2}\sum_{i,j=1}^{N} \sigma\big(C(z_i,t)\cap C(z_j,t)\big) - \big(\sigma\big(C(p,t)\big)\big)^2 .$$
- Taking $t=0$ (i.e. hemispheres): $C(z_i,t)\cap C(z_j,t) = \frac{1}{2}\big(1-d(x,y)\big)$, where d is the normalized geodesic distance $d(x,y)=\frac{1}{\pi}\cos^{-1}(x\cdot y)$.

Hemisphere discrepancy

- L^2 discrepancy for spherical cap discrepancy of fixed height t satisfies:
$$\left[D^{(t)}_{L^2,\text{cap}}(Z)\right]^2 = \frac{1}{N^2}\sum_{i,j=1}^{N} \sigma\big(C(z_i,t)\cap C(z_j,t)\big) - \big(\sigma\big(C(p,t)\big)\big)^2 .$$
- Taking $t=0$ (i.e. hemispheres): $C(z_i,t)\cap C(z_j,t) = \frac{1}{2}\big(1-d(x,y)\big)$, where d is the normalized geodesic distance $d(x,y)=\frac{1}{\pi}\cos^{-1}(x\cdot y)$.

Theorem (Stolarsky for hemispheres, DB '16, Skriganov '16)

$$[D_{L^2,\text{hem}}(Z)]^2 = [D^{(0)}_{L^2,\text{cap}}(Z)]^2$$
$$= \frac{1}{2}\left(\int_{\mathbb{S}^d}\int_{\mathbb{S}^d} d(x,y)\,d\sigma(x)\,d\sigma(y) - \frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i,z_j)\right).$$

Hemisphere Stolarsky: simple corollaries

$$[D_{L^2,\mathrm{hem}}(Z)]^2 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j)\right).$$

Hemisphere Stolarsky: simple corollaries

$$[D_{L^2,\mathrm{hem}}(Z)]^2 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j)\right).$$

- For any $Z = \{z_1, ..., z_N\} \subset \mathbb{S}^d$

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j) \le \frac{1}{2}$$

Hemisphere Stolarsky: simple corollaries

$$[D_{L^2,\text{hem}}(Z)]^2 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j)\right).$$

- For any $Z = \{z_1, ..., z_N\} \subset \mathbb{S}^d$

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j) \leq \frac{1}{2}$$

- For even N:

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j) = \frac{1}{2} \Longleftrightarrow Z \text{ - symmetric.}$$

Hemisphere Stolarsky: simple corollaries

$$[D_{L^2,\mathrm{hem}}(Z)]^2 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j)\right).$$

Hemisphere Stolarsky: simple corollaries

$$[D_{L^2,\mathrm{hem}}(Z)]^2 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j)\right).$$

- For odd N the maximal value is

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j) = \frac{1}{2} - \frac{1}{2N^2}.$$

Hemisphere Stolarsky: simple corollaries

$$[D_{L^2,\text{hem}}(Z)]^2 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j)\right).$$

- For odd N the maximal value is

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j) = \frac{1}{2} - \frac{1}{2N^2}.$$

- It is achieved iff $Z = Z_1 \cup Z_2$, where
  - Z_1 is symmetric,
  - Z_2 lies on a two-dimensional hyperplane (on a great circle) and is a maximizer for $\mathbb{S}^1$.

$$[D_{L^2,\mathrm{hem}}(Z)]^2 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j)\right).$$

- For odd N the maximal value is

$$\frac{1}{N^2}\sum_{i,j=1}^{N} d(z_i, z_j) = \frac{1}{2} - \frac{1}{2N^2}.$$

- It is achieved iff $Z = Z_1 \cup Z_2$, where
  - Z_1 is symmetric,
  - Z_2 lies on a two-dimensional hyperplane (on a great circle) and is a maximizer for $\mathbb{S}^1$.

- Fejes-Toth '59: $d = 1$ and conjectured for $d \geq 2$.
- Sperling, '60 ($d = 2$, even N)
- Larcher, '61 ($d = 2$, odd N)

Hemisphere Stolarsky for general measures

Let μ be a probability measure on $\mathbb{S}^d$. Define the geodesic distance energy integral

$$I_g(\mu) = \int\limits_{\mathbb{S}^d}\int\limits_{\mathbb{S}^d} d(x,y)\, d\mu(x) d\mu(y).$$

Hemisphere Stolarsky for general measures

Let μ be a probability measure on $\mathbb{S}^d$. Define the geodesic distance energy integral

$$I_g(\mu) = \int\limits_{\mathbb{S}^d}\int\limits_{\mathbb{S}^d} d(x,y)\, d\mu(x) d\mu(y).$$

Let $H(x) = C(x,0)$ denote the hemisphere with center at x. Then the following version of the Stolarsky principle holds:

$$\int\limits_{\mathbb{S}^d} \left(\mu\big(H(x)\big) - \frac{1}{2} \right)^2 d\sigma(x) = \frac{1}{2} \cdot \left(\frac{1}{2} - I_g(\mu) \right).$$

Hemisphere Stolarsky for general measures

Let μ be a probability measure on $\mathbb{S}^d$. Define the geodesic distance energy integral

$$I_g(\mu) = \int\limits_{\mathbb{S}^d}\int\limits_{\mathbb{S}^d} d(x,y)\, d\mu(x) d\mu(y).$$

Let $H(x) = C(x,0)$ denote the hemisphere with center at x. Then the following version of the Stolarsky principle holds:

$$\int\limits_{\mathbb{S}^d} \left(\mu\big(H(x)\big) - \frac{1}{2}\right)^2 d\sigma(x) = \frac{1}{2}\cdot\left(\frac{1}{2} - I_g(\mu)\right).$$

- For any probability measure μ: $\quad I_g(\mu) \le \frac{1}{2}$.

Hemisphere Stolarsky for general measures

Let μ be a probability measure on $\mathbb{S}^d$. Define the geodesic distance energy integral

$$I_g(\mu) = \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} d(x,y)\, d\mu(x) d\mu(y).$$

Let $H(x) = C(x,0)$ denote the hemisphere with center at x. Then the following version of the Stolarsky principle holds:

$$\int_{\mathbb{S}^d} \left(\mu\big(H(x)\big) - \frac{1}{2}\right)^2 d\sigma(x) = \frac{1}{2} \cdot \left(\frac{1}{2} - I_g(\mu)\right).$$

- For any probability measure μ: $\quad I_g(\mu) \leq \frac{1}{2}$.
- $I_g(\mu) = \frac{1}{2}$ (i.e. μ is a maximizer) iff $\mu(H(x)) = \frac{1}{2}$ for σ-a.e. $x \in \mathbb{S}^d$

Hemisphere Stolarsky for general measures

Let μ be a probability measure on $\mathbb{S}^d$. Define the geodesic distance energy integral

$$I_g(\mu) = \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} d(x,y)\, d\mu(x) d\mu(y).$$

Let $H(x) = C(x,0)$ denote the hemisphere with center at x. Then the following version of the Stolarsky principle holds:

$$\int_{\mathbb{S}^d} \left(\mu(H(x)) - \frac{1}{2}\right)^2 d\sigma(x) = \frac{1}{2} \cdot \left(\frac{1}{2} - I_g(\mu)\right).$$

- For any probability measure μ: $\quad I_g(\mu) \leq \frac{1}{2}$.
- $I_g(\mu) = \frac{1}{2}$ (i.e. μ is a maximizer) iff
  $\mu(H(x)) = \frac{1}{2}$ for σ-a.e. $x \in \mathbb{S}^d$ iff
  μ is symmetric, i.e. $\mu(E) = \mu(-E)$.

Let μ be a Borel probability measure on $\mathbb{S}^d$.
Then

$$I_E(\mu) = \int\limits_{\mathbb{S}^d} \int\limits_{\mathbb{S}^d} |x - y| \, d\mu(x) d\mu(y)$$

has a unique maximizer $\mu = \sigma$ (Bjorck, '56)

Distance energy integrals

Let μ be a Borel probability measure on $\mathbb{S}^d$.
Then

$$I_E(\mu) = \int\limits_{\mathbb{S}^d}\int\limits_{\mathbb{S}^d} |x-y| \, d\mu(x) d\mu(y)$$

has a unique maximizer $\mu = \sigma$ (Bjorck, '56)

However,

$$I_g(\mu) = \int\limits_{\mathbb{S}^d}\int\limits_{\mathbb{S}^d} d(x,y) \, d\mu(x) d\mu(y)$$

is maximized by any symmetric measure μ.

Euclidean distance energy integrals

Let μ be a Borel probability measure on the sphere $\mathbb{S}^d$. For $\lambda > 0$ define the energy integral

$$I_\lambda = \int\limits_{\mathbb{S}^d}\int\limits_{\mathbb{S}^d} |x-y|^\lambda d\mu(x)d\mu(y)$$

Maximizers (Bjorck '56):

- $0 < \lambda < 2$: unique maximizer is surface measure σ,
- $\lambda = 2$: any measure with center of mass at 0,
- $\lambda > 2$: mass $\frac{1}{2}$ at two opposite poles.

Riesz energy

- $-d < \lambda \leq 0$: unique minimizer is surface measure σ

Geodesic distance energy integrals

Let μ be a Borel probability measure on the sphere $\mathbb{S}^d$. For $\lambda > 0$ define the energy integral

$$I_\lambda = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \big(d(x,y)\big)^\lambda d\mu(x) d\mu(y)$$

Maximizers (DB, Dai, Matzke '16):

- $-d < \lambda \leq 0$: unique minimizer is σ
- $0 < \lambda < 1$: unique maximizer is σ,
- $\lambda = 1$: any symmetric measure,
- $\lambda > 1$: mass $\frac{1}{2}$ at two opposite poles.

Let μ be a Borel probability measure on the sphere $\mathbb{S}^d$. For $\lambda > 0$ define the energy integral

$$I_\lambda = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \big(d(x,y)\big)^\lambda d\mu(x) d\mu(y)$$

Maximizers (DB, Dai, Matzke '16):

- $-d < \lambda \leq 0$: unique minimizer is σ
- $0 < \lambda < 1$: unique maximizer is σ,
- $\lambda = 1$: any symmetric measure,
- $\lambda > 1$: mass $\frac{1}{2}$ at two opposite poles.

$d = 1$: Brauchart, Hardin, Saff, '12

Positive definite functions on the sphere

Consider the energy integral

$$I_F(\mu) = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} F(x \cdot y) d\mu(x) d\mu(y).$$

Positive definite functions on the sphere

Consider the energy integral

$$I_F(\mu) = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} F(x \cdot y) d\mu(x) d\mu(y).$$

A function $F \in C[-1,1]$ is called *positive definite* on the sphere $\mathbb{S}^d$ if for any set of points $Z = \{z_1, ..., z_N\} \subset \mathbb{S}^d$, the matrix $\big[F(z_i \cdot z_j)\big]_{i,j=1}^N$ is positive semidefinite, i.e.

$$\sum_{i,j} F(z_i \cdot z_j) c_i c_j \geq 0 \quad \text{for all } c_i \in \mathbb{R}.$$

Positive definite functions on the sphere

Lemma

For a function $F \in C[-1,1]$ the following are equivalent:

(i) *F is positive definite on $\mathbb{S}^d$.*

(ii) *Gegenbauer coefficients of F are non-negative, i.e.*

$$\widehat{F}(n,\lambda) \geq 0 \text{ for all } n \geq 0.$$

(iii) *For any signed measure $\mu \in \mathcal{B}$ the energy integral is non-negative: $I_F(\mu) \geq 0$.*

(iv) *There exists a function $f \in L^2_{w_\lambda}[-1,1]$ such that*

$$F(x \cdot y) = \int_{\mathbb{S}^d} f(x \cdot z) f(z \cdot y)\, d\sigma(z), \quad x, y \in \mathbb{S}^d,$$

i.e. F is the spherical convolution of f with itself.
$\widehat{f}(n,\lambda)^2 = \widehat{F}(n,\lambda)$

Theorem (DB, F. Dai, '16)

- *Generalized Stolarsky principle:*

$$I_F(\mu) - I_F(\sigma) = D^2_{L^2,f}(\mu).$$

Generalized Stolarsky principle

Theorem (DB, F. Dai, '16)

- *Generalized Stolarsky principle:*

$$I_F(\mu) - I_F(\sigma) = D^2_{L^2,f}(\mu).$$

- *Estimates for optimal N-point discrepancy:*

$$\min_{1 \le k \lesssim N^{1/d}} \widehat{F}(k,\lambda) \lesssim \mathcal{D}^2_{L^2,f,N} \lesssim \frac{1}{N} \max_{0 \le \theta \lesssim N^{-\frac{1}{d}}} \big(F(1) - F(\cos\theta)\big)$$

Asymptotics of optimal discrete geodesic energy

Let

$$\mathcal{E}_{d,\delta} = \inf_{Z_N} \sum_{1 \le i < j \le N} \big(d(z_i, z_j)\big)^{\delta},$$

$$I_{d,\delta}(\sigma) = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \big(d(x,y)\big)^{\delta} d\sigma(x) d\sigma(y).$$

Theorem (DB, Dai '16)

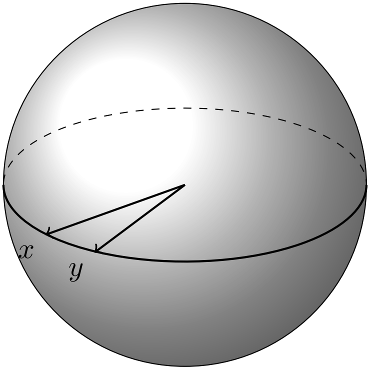
- *If* $-d < \delta < 1$ *and* $\delta \neq 0$, *then*

$$I_{d,\delta}(\sigma) - \frac{2}{N^2} \mathcal{E}_{d,\delta}(N) \sim N^{-1-\frac{\delta}{d}}$$

- *In the logarithmic case,* $\delta = 0$,

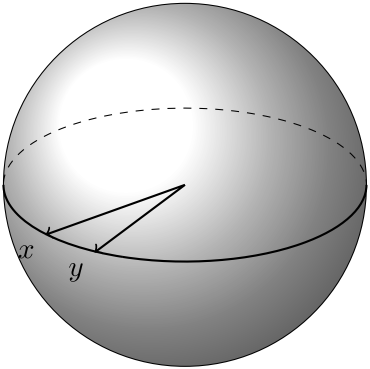
$$I_{d,0}(\sigma) - \frac{2}{N^2} \mathcal{E}_{d,0}(N) \sim N^{-1} \log N.$$

In the Euclidean case: Wagner, Saff–Kuijlaars, Brauchart



Let x, $y \in \mathbb{S}^d$
choose a random hyperplane $z^{\perp}$, $z \in \mathbb{S}^d$.

Tessellations of spheres (joint work with Michael Lacey)



Let x, $y \in \mathbb{S}^d$
choose a random hyperplane $z^\perp$, $z \in \mathbb{S}^d$.

Then

$$\begin{aligned} &\mathbb{P}(z^\perp \text{ separates } x \text{ and } y) \\ &= \mathbb{P}(\operatorname{sign}\langle z, x\rangle \neq \operatorname{sign}\langle z, y\rangle) \\ &= d(x, y), \end{aligned}$$

where d is the normalized geodesic distance on the sphere, i.e. $d(x, y) = \frac{\cos^{-1}\langle x,y\rangle}{\pi}$.

Hamming distance

Consider a set of vectors $Z = \{z_1, z_2, ..., z_N\}$ on the sphere $\mathbb{S}^d$. Define the Hamming distance as

$$d_H(x, y) := \frac{\#\big\{z_k \in Z : \operatorname{sign}(x \cdot z_k) \neq \operatorname{sign}(y \cdot z_k)\big\}}{N},$$

i.e. the proportion of hyperplanes $z_k^\perp$ that *separate* x and y.

The main question

Define

$$\Delta_Z(x,y) := d_H(x,y) - d(x,y).$$

The main question

Define

$$\Delta_Z(x,y) := d_H(x,y) - d(x,y).$$

Let $K \subset \mathbb{S}^d$. We say that Z *induces a δ-uniform tessellation of* K if

$$\sup_{x,y\in K} \left|\Delta_Z(x,y)\right| \leq \delta.$$

Define

$$\Delta_Z(x,y) := d_H(x,y) - d(x,y).$$

Let $K \subset \mathbb{S}^d$. We say that Z *induces a δ-uniform tessellation of K* if

$$\sup_{x,y\in K} \left|\Delta_Z(x,y)\right| \le \delta.$$

Equivalently, the *sign-linear* map $\phi_Z(x) = \{\text{sign}(x \cdot z_k)\}_{k=1}^N = \text{sign}(Zx)$

is a δ-isometry from K into the Hamming cube $\mathcal{H}^N = \{-1,1\}^N$.

The main question

Question: Given $K \subset \mathbb{S}^d$ and $\delta > 0$, what is the smallest value of N so that K can be δ-isometrically embedded into $\mathcal{H}^N$?

The main question

Question: Given $K \subset \mathbb{S}^d$ and $\delta > 0$, what is the smallest value of N so that K can be δ-isometrically embedded into $\mathcal{H}^N$?

Examples of K:

- $K = \mathbb{S}^d$
- K finite
- sparse vectors (one-bit compressed sensing)

The main question

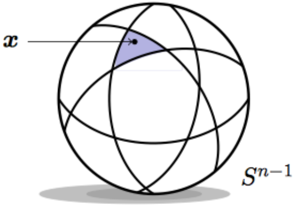
Question: Given $K \subset \mathbb{S}^d$ and $\delta > 0$, what is the smallest value of N so that K can be δ-isometrically embedded into $\mathcal{H}^N$?

Examples of K:

- $K = \mathbb{S}^d$
- K finite
- sparse vectors (one-bit compressed sensing)

Prior results:
Plan, Vershynin, '13: $N = C\delta^{-6}\omega(K)^2$ random points yield a δ-uniform tessellation of K with high probability.

Cells with small diameter

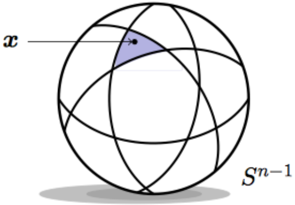


Picture from Baraniuk, Foucart, Needell, Plan, Wooters

Lemma

Every cell of a δ-uniform tessellation of K by hyperplanes has diameter at most δ.

Cells with small diameter



Picture from Baraniuk, Foucart, Needell, Plan, Wooters

Lemma

Every cell of a δ-uniform tessellation of K by hyperplanes has diameter at most δ.

Proof:
if x and y are in the same cell then

$$d(x,y) = \left|d(x,y) - \underbrace{d_H(x,y)}_{=0}\right| \leq \delta.$$

Main results (DB, Lacey, '16)

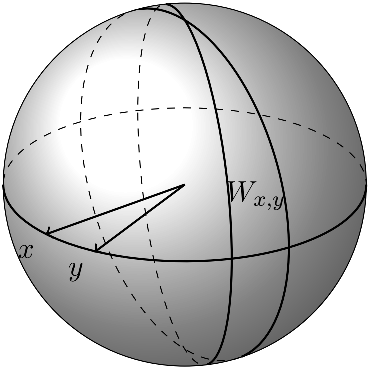
- **Small cells:** If $N \gtrsim \delta^{-1} \log N(K, c\delta)$, then w.h.p. N random vectors induce a tessellation with δ-small cells.

- **Uniform tessellation:** If $N \gtrsim \delta^{-2} H(K)^2$, then there exists a δ-isometry $\phi : \mathbb{S}^d \to \mathcal{H}^N$.

- **Sparse case:** Let K_s be the set of s-sparse vectors in $\mathbb{S}^d$. If $N \gtrsim \delta^{-2} s \log_+ \frac{d}{s}$, then for a random set Z of m points in $\mathbb{S}^d$ w.h.p. the *sign-linear map* is a δ-isometry from K_s to $\mathcal{H}^N$.

- **One-bit Johnson-Lindenstrauss lemma:** If K is finite and $m \gtrsim \delta^{-2} \log(\#K)$, then there exists a δ-isometry between $K \subset \mathbb{S}^d$ and the Hamming cube $\mathcal{H}^m$.

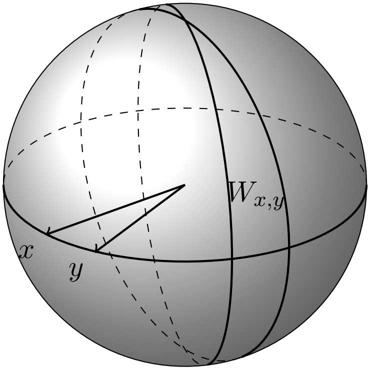
Tessellations and discrepancy



$$H_x = \{z \;:\; \langle z, x\rangle > 0\}$$

$$W_{xy} = H_x \triangle H_y$$
$$= \{z \in \mathbb{S}^d : \operatorname{sign}\langle z, x\rangle \neq \operatorname{sign}\langle z, y\rangle\}$$

Tessellations and discrepancy

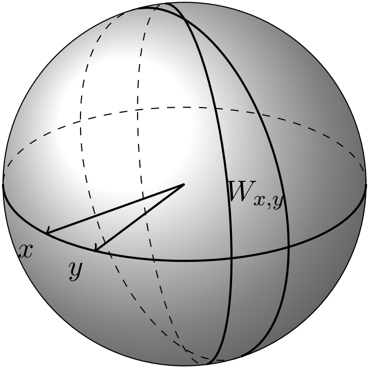


$$H_x = \{z \ : \ \langle z, x\rangle > 0\}$$

$$W_{xy} = H_x \triangle H_y$$
$$= \{z \in \mathbb{S}^d : \operatorname{sign}\langle z, x\rangle \neq \operatorname{sign}\langle z, y\rangle\}$$

$$\mathbb{P}(\operatorname{sign}\langle z, x\rangle \neq \operatorname{sign}\langle z, y\rangle)$$
$$= \sigma(W_{xy}) = d(x, y)$$

Tessellations and discrepancy



$$H_x = \{z \; : \; \langle z, x\rangle > 0\}$$

$$W_{xy} = H_x \triangle H_y$$
$$= \{z \in \mathbb{S}^d : \operatorname{sign}\langle z, x\rangle \neq \operatorname{sign}\langle z, y\rangle\}$$

$$\mathbb{P}(\operatorname{sign}\langle z, x\rangle \neq \operatorname{sign}\langle z, y\rangle)$$
$$= \sigma(W_{xy}) = d(x, y)$$

$$\Delta_Z(x,y) = d_H(x,y) - d(x,y) = \frac{\#(Z \cap W_{xy})}{N} - \sigma(W_{xy})$$

$$D_{\text{wedge}}(Z) = \left\| \Delta_Z(x,y) \right\|_\infty = \sup_{x,y \in \mathbb{S}^d} \left| \frac{\#(Z \cap W_{xy})}{N} - \sigma(W_{xy}) \right|.$$

Tessellations and discrepancy (DB, Lacey, '16)

- There exist constants c_d, C_d, such that the following discrepancy bounds hold:

$$c_d N^{-\frac{1}{2}-\frac{1}{2d}} \leq \inf_{Z\subset\mathbb{S}^d:\,\#Z=N} D_{\text{wedge}}(Z) \leq C_d N^{-\frac{1}{2}-\frac{1}{2d}}\sqrt{\log N}.$$

Tessellations and discrepancy (DB, Lacey, '16)

- There exist constants c_d, C_d, such that the following discrepancy bounds hold:

$$c_d N^{-\frac{1}{2}-\frac{1}{2d}} \leq \inf_{Z\subset\mathbb{S}^d:\,\#Z=N} D_{\text{wedge}}(Z) \leq C_d N^{-\frac{1}{2}-\frac{1}{2d}}\sqrt{\log N}.$$

- Inverting this we find that the optimal value of N satisfies

$$\delta^{-2+\frac{2}{d+1}} \lesssim N \lesssim \delta^{-2+\frac{2}{d+1}}\left(\log\frac{1}{\delta}\right)^{\frac{d}{d+1}}.$$

Stolarsky for wedge discrepancy (DB, Lacey, '16)

Define the L^2 discrepancy for wedges

$$[D_{L^2,\text{wedge}}(Z)]^2 = \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} \left(\frac{1}{N}\sum_{k=1}^{N} \mathbf{1}_{W_{xy}}(z_k) - \sigma(W_{xy}) \right)^2 d\sigma(x)\, d\sigma(y)$$

Stolarsky for wedge discrepancy (DB, Lacey, '16)

Define the L^2 discrepancy for wedges

$$[D_{L^2,\text{wedge}}(Z)]^2 = \int\limits_{\mathbb{S}^d}\int\limits_{\mathbb{S}^d} \left(\frac{1}{N}\sum_{k=1}^{N} \mathbf{1}_{W_{xy}}(z_k) - \sigma(W_{xy}) \right)^2 d\sigma(x)\, d\sigma(y)$$

Theorem (Stolarsky for wedges, DB, Lacey, '15)

For any finite set $Z = \{z_1, \dots, z_N\} \subset \mathbb{S}^d$

$$[D_{L^2,\text{wedge}}(Z)]^2 = \frac{1}{N^2}\sum_{i,j=1}^{N} \left(\frac{1}{2} - d(z_i, z_j)\right)^2 - \int\limits_{\mathbb{S}^d}\int\limits_{\mathbb{S}^d} \left(\frac{1}{2} - d(x,y)\right)^2 d\sigma(x)\, d\sigma(y).$$

Frame potential

- $Z = \{z_1, \dots, z_N\} \subset \mathbb{S}^d$ is a frame in $\mathbb{R}^d$ iff there exist $c, C > 0$ such that for any $x \in \mathbb{R}^{d+1}$

$$c\|x\|^2 \leq \sum_k |\langle x, z_k \rangle|^2 \leq C\|x\|^2.$$

Frame potential

- $Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d$ is a frame in $\mathbb{R}^d$ iff there exist $c, C > 0$ such that for any $x \in \mathbb{R}^{d+1}$

$$c\|x\|^2 \leq \sum_k |\langle x, z_k \rangle|^2 \leq C\|x\|^2.$$

- $Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d$ is a tight frame iff there exists $A > 0$ such that for any $x \in \mathbb{R}^{d+1}$

$$\sum_k |\langle x, z_k \rangle|^2 = A\|x\|^2.$$

Frame potential

- $Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d$ is a frame in $\mathbb{R}^d$ iff there exist $c, C > 0$ such that for any $x \in \mathbb{R}^{d+1}$

$$c\|x\|^2 \leq \sum_k |\langle x, z_k \rangle|^2 \leq C\|x\|^2.$$

- $Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d$ is a tight frame iff there exists $A > 0$ such that for any $x \in \mathbb{R}^{d+1}$

$$\sum_k |\langle x, z_k \rangle|^2 = A\|x\|^2.$$

Theorem (Benedetto, Fickus)

A set $Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d$ is a tight frame in $\mathbb{R}^{d+1}$ if and only if Z is a local minimizer of the frame potential:

$$F(Z) = \sum_{i,j=1}^{N} |\langle z_i, z_j \rangle|^2.$$