Thoughts on the Coulomb Plasma

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Particle systems

A system $\{\zeta_i\}_1^n \in \mathbb{C}$ ("point charges") in external field nQ.

- **Energy:**

$$H_n = \sum_{j\neq k}^{n} \log \frac{1}{|\zeta_j - \zeta_k|} + n \sum_{j=1}^{n} Q(\zeta_j).$$

- **Boltzmann–Gibbs law:**

$$d\mathbf{P}_n(\zeta) = \frac{1}{Z_n^{\beta}} e^{-\beta H_n(\zeta)} \, d^{2n}\zeta, \quad \zeta = (\zeta_j)_1^n. \tag{1}$$

- **Assumptions.** $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ is l.s.c., C^{ω}-smooth, and

$$Q(\zeta) >> \log |\zeta|, \quad (\zeta \to \infty).$$

A minimizer $\{\zeta_j\}_1^n$ of H_n is a *Fekete-configuration*.

Frostman's equilibrium measure

- *Q-energy* of a Borel p.m. μ on $\mathbb{C}$

$$I(\mu) := \iint \log \frac{1}{|\zeta - \eta|}\, d\mu(\zeta) d\mu(\eta) + \int Q\, d\mu.$$

  The *equilibrium measure* σ minimizes $I(\mu)$: μ p.m.

- *Droplet*

$$S = S[Q] := \operatorname{supp} \sigma. \tag{2}$$

Frostman:

$$d\sigma(z) = \chi_S(z)\, \Delta Q(z)\, dA(z).$$

Large deviation estimate: if $\{\zeta_j\}_1^n$ random sample, f continuous, bounded,

$$\frac{1}{n} \mathbf{E}_n^{(\beta)}(f(\zeta_1) + \ldots + f(\zeta_n)) \to \sigma(f).$$

Example: Ginibre ensemble ($\beta = 1$)

Let $Q(\zeta) = |\zeta|^2$. Then $S = \{|\zeta| \le 1\}$ and $\sigma = \chi_S \, dA$.
The process $\{\zeta_i\}_1^n$ can be interpreted as eigenvalues of an $n \times n$-matrix with i.i.d. centered complex Gaussian entries of variance $1/n$.

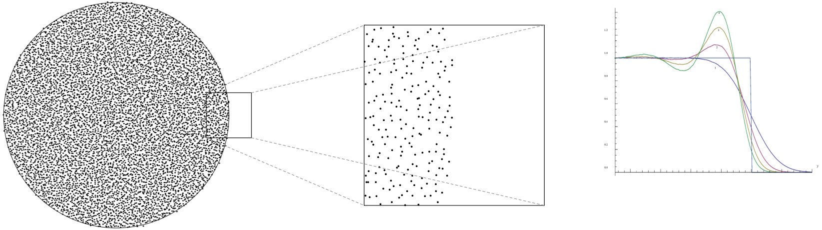


Figure : Left: Ginibre particles for $\beta = 1$. Right: boundary profiles for $\beta = 1, 2, 3, 4$

Sakai theory

Technical assumptions:

- Q is real-analytic in a nbh of S.
- $\Delta Q > 0$ in a nbh of ∂S.

Conclusions:

- S^c is an *Unbounded Quadrature Domain* (in wide sense of Shapiro).
- ∂S is a union of finitely many analytic curves. Possible singularities: cusps pointing out of S and double points.

Droplets 1

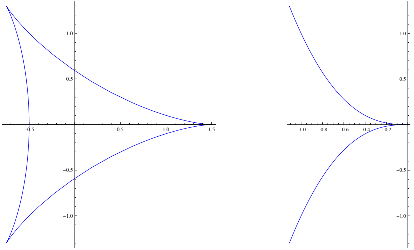
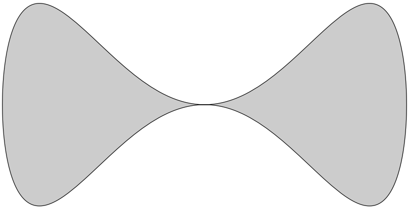


Figure : The Deltoid is not admissible; it has three maximal 3/2 cusps. 5/2 cusp is OK.

Droplets 2



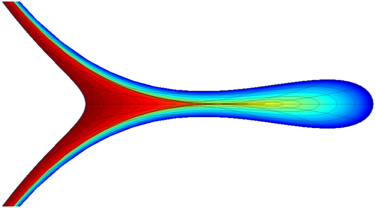


Figure : Double point and 5/2 cusp under Laplacian growth.

Global results

Linear statistics on $(\mathbb{C}^n, \mathbf{P}_n)$

$$\mathrm{fluct}_n(f) = \sum_1^n f(\zeta_j) - n\sigma(f), \quad (f \in C_b^\infty(\mathbb{C})).$$

$\mathrm{fluct}_n(f)$ converges in distribution to the normal $N(e_f, \sigma_f^2)$, where

$$e_f = (\frac{1}{\beta} - \frac{1}{2}) \int_{\mathbb{C}} f \cdot \Delta(\chi_S + L^S), \quad \sigma_f^2 = \int_{\mathbb{C}} |\partial f^S|^2, \quad (L = \log \Delta Q).$$

Here f^S equals f in S and is harmonic and bounded in S^c.

- $\beta = 1$, ∂S C^1-smooth, S connected, f C^2-smooth. (MAH 2011)
- $\beta > 0$, f smooth, supported in the bulk. (BBNY 2016)

Intensity functions

Let $N_\epsilon(\eta)$ number of ζ_j which hit $D(\eta,\epsilon)$.
1-point function:

$$\mathbf{R}_n(\eta) = \lim_{\epsilon\to 0} \frac{\mathbf{E}_n^{(\beta)}(N_\epsilon(\eta))}{\epsilon^2}.$$

2-point function:

$$\mathbf{R}_{n,2}(\eta_1,\eta_2) = \lim_{\epsilon\to 0} \frac{\mathbf{E}_n^{(\beta)}(N_\epsilon(\eta_1)\cdot N_\epsilon(\eta_2))}{\epsilon^4}.$$

If $\beta = 1$, the process is *determinantal*,

$$\mathbf{R}_{n,k}(\eta_1,\ldots,\eta_k) = \det\left(\mathbf{K}_n(\eta_i,\eta_j)\right)_{i,j=1}^{k}.$$

Here $\mathbf{K}_n$ is a "correlation kernel” = reprokernel for

$$\mathcal{W}_n := \{q\cdot e^{-nQ/2};\ \text{degree}(q) < n\} \subset L^2.$$

Ward's identity

Let $\{\zeta_j\}_1^n$ system. For smooth ψ define r.v.'s

$$A_\psi = \frac{1}{2}\sum_{j\neq k}^{n} \frac{\psi(\zeta_j)-\psi(\zeta_k)}{\zeta_j-\zeta_k},\ B_\psi = n\sum_{1}^{n} \partial Q(\zeta_j)\psi(\zeta_j),\ C_\psi = \sum_{1}^{n} \partial\psi(\zeta_j).$$

Theorem

For all ψ

$$\mathbf{E}_n(\beta\cdot(A_\psi - B_\psi) + C_\psi) = 0.$$

This is an implicit relation between $\mathbf{R}_n$ and $\mathbf{R}_{n,2}$.

(Proof: reparametrization invariance of the partition function $Z_n := \int_{\mathbb{C}^n} e^{-\beta H_n}\,dV_n$.)

Microscopic scale

Fix $p \in S$. $r_n = r_n(p)$ satisfies:

$$n \cdot \int_{D(p_n, r_n)} \Delta Q(\zeta)\, dA(\zeta) = 1.$$

- Regular case: If $\Delta Q(p) > 0$ then

$$r_n \sim \frac{1}{\sqrt{n \Delta Q(p)}}.$$

- Vanishing equilibrium density to order k: If k is smallest s.t. $\Delta^k Q(p) > 0$ then

$$r_n \sim \left(\frac{k[(k-1)!]^2}{\Delta^k Q(p)}\right)^{1/2k} \cdot n^{-1/2k}.$$

Rescaled system

$z_j = r_n(\zeta_j - p)$.

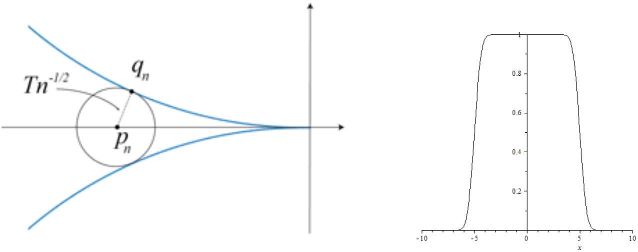


Figure : Left: a moving point p_n approaching a cusp. Right: the profile of a translation invariant "candidate" for the micro-density at p_n, $\beta = 1$.

Rescaling at bulk singularities

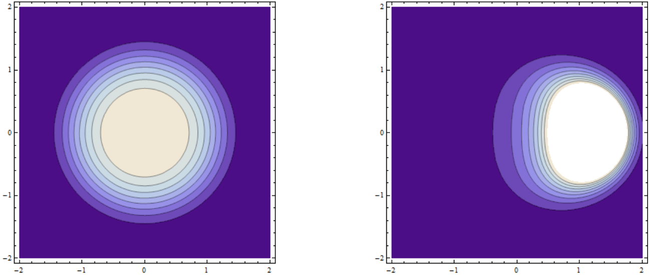


Figure : These figures show the repelling effect of inserting a point charge close to a bulk singularity caused by vanishing equilibrium density.

Free boundary vs hard edge

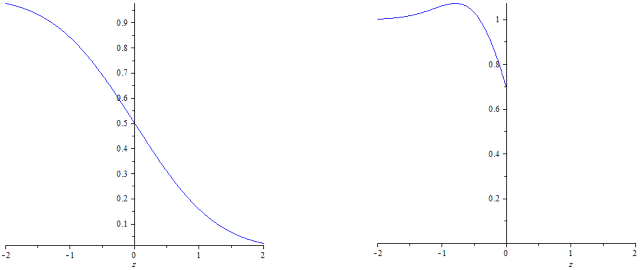


Figure : The hard edge is obtained by redefining $Q = +\infty$ outside S. The intensity has been computed for $\beta = 1$.

- Free boundary $\leftrightarrow$ GFF with *free* BCs.
- Hard edge $\leftrightarrow$ GFF with *Neumann* BCs. (Joint w/ H.-J. Tak.)

Gaussian field with Dirichlet BCs in the disk

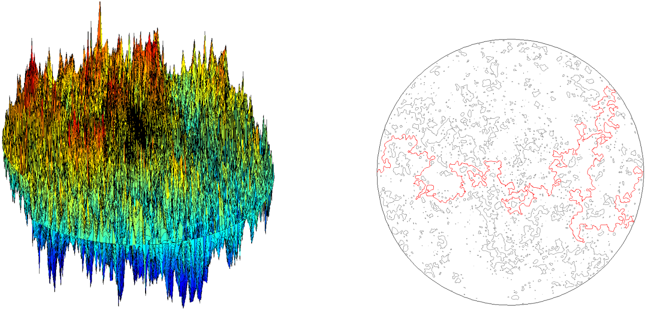


Figure : A field approximation Φ_n. The figure on the right shows the level curve $\Phi_n + h = 1/2$ where h is harmonic measure for the upper half-circle. The level curve resembles an SLE_4, in accordance with Sheffield-Schramm's theorem.

Three relevant BCs: Dirichlet, Free, Neumann.

Ward's equation

- $R_n(z) = \mathbf{R}_n(\zeta)$
- $R_{n,2}(z,w) = \mathbf{R}_{n,2}(\zeta,\eta)$, $z = r_n(\zeta - p)$, $w = r_n(\eta - p)$.
- $B_n(z,w) = (R_n(z)R_n(w) - R_{n,2}(z,w))/R_n(z)$.
- $C_n(z) := \int \frac{B(z,w)}{z-w}\, dA(w)$.
- Ward's equation:

$$\bar{\partial} C_n(z) = R_n(z) - 1 - \frac{1}{\beta}\Delta \log R_n(z) + o(1).$$

If $\beta = 1$ then normal families show $R_{n_k} \to R$, $C_{n_k} \to C = C$, where $R \to C$ by analytic continuation. So

$$\bar{\partial} C = R - 1 - \Delta \log R$$

is an equation for the single function R.

Translation invariance

- To find a true micro-density, we need side-conditions in Ward's equation.
- It is natural to assume *translation invariance*: $R(z) = F(z + \bar{z})$ for some function F.
- The complete t.i. solution to Ward's equation was given in AKM 14.

The above might give a "physical proof" of universality, but for a mathematical proof we must rule out the possibility of non-t.i. solutions.

- For t.i. solutions, Ward's equation can be written as a convolution equation and solved by Fourier analysis.
- For possibly non-t.i. solutions, we get a *twisted convolution equation*, known from Fourier analysis on the Heisenberg group. (Joint w/ J.-L. Romero.)

Spacings

Fix $0 \in S$ and let $z_j = r_n^{-1}\zeta_j$, $j = 1, \dots, n$.
Put

$$F_n = \{\text{at least one particle falls in } \mathbb{D}\}, \quad \eta_n = \mathbf{P}_n^{(\beta)}(F_n).$$

Spacing at 0:

$$s_0 = \min_{z_j \in \mathbb{D}} \min_{k \neq j} |z_j - z_k|, \quad \{z_j\}_1^n \in F_n.$$

Repulsion theorem: if $\beta > 1$ then there is a constant $c = c(n, \beta) > 0$ so that

$$\mathbf{P}_n^{(\beta)}(\{s_0 \geq c \cdot (\epsilon\eta_n)^{\frac{1}{2(\beta-1)}} | F_n) \geq 1 - m_0\epsilon, \quad 0 < \epsilon < 1,$$

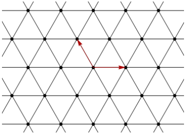
where $m_0 = 16c^{-2}(\epsilon\eta_n)^{-\frac{1}{\beta-1}}$.
Proof: (i) Estimate expected $L^{2\beta}$-norm for weighted random Lagrange polynomials, (ii) Use Bernstein to estimate expected $L^{2\beta}$ norm of gradient, (iii) Morrey's and Chebyshev's inequalities give estimate for distance between zeros, with high probability.

Crystallization

Corollary: if $c < 1/(8\sqrt{e})$ and $n = n(c)$ large enough, then

$$\lim_{\beta\to\infty} \mathbf{P}_n^{(\beta)}(\{s_0 > c\}) = 1.$$

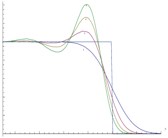
Abrikosov conjecture: the right bound should be $c < 2^{1/2}3^{-1/4}$.



Q: What patterns will emerge near a bulk singularity caused by vanishing equilibrium density?

Hall effect

When $\beta > 1$ the intensity R has some "irregularities" near the boundary:



- We believe that a micro-density R is translation invariant and solves Ward: $\partial\bar{C} = R - 1 - \frac{1}{\beta}\Delta \log R$.
- Q: Is there a critical "freezing temperature" $1/\beta_0$ after which crystallization takes place?

Conical singularities

Fix $c < 1$. n-dependent potential

$$V_n(\zeta) = Q(\zeta) + (c/n)G_0(\zeta)$$

where G_0 is Green's function for S with pole at 0.
Micro-scale:

$$c + \int_{D(0,r_n)} \Delta Q \, dA = 1.$$

Rescale: $z_j = r_n^{-1}\zeta_j$ and let R_n be 1-point function of $\{\zeta_j\}_{j=1}^n$.
Normal families: $R_{n_k} \to R$ as distributions on $\mathbb{C}$, locally uniformly on $\dot{\mathbb{C}}$.
Microscopic potential:

$$V_0(z) = Q_0(z) + 2c \log|z|$$

where Q_0 is the dominant part of the Taylor expansion of Q about 0.

Fock-Sobolev spaces

Let $\mu_0 = e^{-V_0} dA$.
Fock-Sobolev space $L^2_a(\mu_0)$ consists of entire functions u s.t. $\int |u|^2 e^{-V_0}\, dA < \infty$.
Let $L_0(z, w)$ Bergman kernel and $R_0(z) = L_0(z, z)$ Bergman function.
Theorem: In many situations, $R = R_0$.

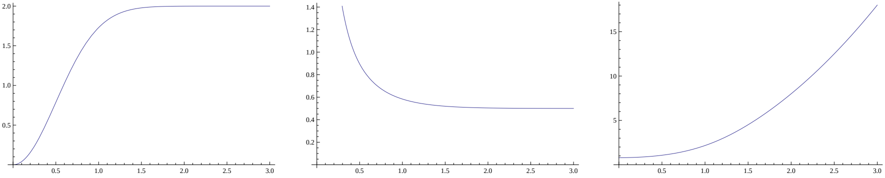


Figure : The Bergman function R_0 as a function of positive reals, for $V_0 = 2|z|^2 - 2\log|z|$, $V_0 = |z|^2/2 + \log|z|$, and $V_0 = |z|^4/2$, respectively.

Conical singularities

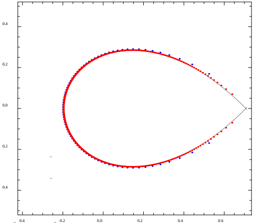
Micro-conformal metric

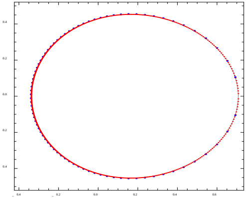
$$ds^2(z) = e^{-V_0(z)}|dz|^2 = |z|^{-2c}e^{-Q_0(z)}|dz|^2.$$

This metric has a conical singularity with total angle $2\pi(1-c)$ at 0, i.e., the Gaussian curvature $\kappa(z) = e^{2V_0(z)}\Delta V_0(z)$ has a singularity there. This is related to *CFT on Riemann surfaces* of Kang-Makarov, Wiegmann et al.

Orthogonal polynomials

Lee and Yang studied OP's for Ginibre ensemble with log-singularity:





The zeros are located on a certain "potential theoretical skeleton", likely to be universal.

ESKERRIK!